

# ENHANCED ADIC FORMALISM AND PERVERSE $t$ -STRUCTURES FOR HIGHER ARTIN STACKS

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**ABSTRACT.** In this sequel of [17, 18], we develop an adic formalism for étale cohomology of Artin stacks and prove several desired properties including the base change theorem. In addition, we define perverse  $t$ -structures on Artin stacks for general perversity, extending Gabber's work on schemes. Our results generalize results of Laszlo and Olsson on adic formalism and middle perversity. We continue to work in the world of  $\infty$ -categories in the sense of Lurie, by enhancing all the derived categories, functors, and natural transformations to the level of  $\infty$ -categories.

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## INTRODUCTION

In [17, 18], we developed a theory of Grothendieck's six operations for étale cohomology of Artin stacks and prove several desired properties including the base change theorem. In the article, we develop the corresponding adic formalism and establish adic analogues of results of [18]. This extends all previous theories on the subject, including SGA 5 [3], Deligne [6], Ekedahl [7] (for schemes), Behrend [4] and Laszlo–Olsson [14]. We prove, among other things, the base change theorem in derived categories, which was previous known only on the level of sheaves [14] (and under other restrictions). Another limitation of the existing theories, including those for schemes, is the constructibility assumption. This assumption is not often met, for example, when considering morphisms between Artin stacks that are only locally of finite type. By contrast, the adic formalism developed in this article applies to unrestricted derived categories.

In addition, we define perverse  $t$ -structures on Artin stacks for general perversity, extending the work of Gabber [9] for schemes and the work of Laszlo and Olsson [15] for the middle perversity.

As in our preceding article, the approach we are taking is different from all the previous theories. We work in the world of  $\infty$ -categories in the sense of Lurie [19, 20], by enhancing all the derived categories, functors, and natural transforms to the level of  $\infty$ -categories. At this level, we may use some new machineries among which the most important ones are gluing objects, Adjoint Functor Theorem,  $\infty$ -categorical descent, all in [19, 20], and some other techniques developed in [18]. In particular, we obtain several other special descent properties for the derived category of lisse-étale sheaves.

**0.1. Results.** In this section, we will state our constructions and results only in the classical setting of Artin stacks on the level of usual derived categories (which are homotopy categories of the derived  $\infty$ -categories), among other simplification. We will provide the precise references of the complete results in later chapters, for higher Deligne–Mumford stacks and higher Artin stacks, stated on the level of stable  $\infty$ -categories. We refer the reader to [18, §0.1] for our convention on Artin stacks.

Let  $\mathcal{X}$  be an Artin stack and let  $\lambda$  be a ringed diagram, that is, a functor from a partially ordered set to the category of unital commutative rings. Recall that  $D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \lambda)$  is the full subcategory of  $D(\mathcal{X}_{\text{lis-ét}}, \lambda)$  spanned by complexes whose cohomology sheaves are all Cartesian. We define in §1.4 a strictly full subcategory  $D(\mathcal{X}, \lambda)_{\text{adic}}$  of  $D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \lambda)$  consisting of *adic complexes*, possessing the property that the inclusion  $D(\mathcal{X}, \lambda)_{\text{adic}} \rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \lambda)$  admits a right adjoint functor  $R_{\mathcal{X}}$ . Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of Artin stacks. We then define operations:

$$\begin{aligned} \underline{f}^*: D(\mathcal{X}, \lambda)_{\text{adic}} &\rightarrow D(\mathcal{Y}, \lambda)_{\text{adic}}, & \underline{f}_*: D(\mathcal{Y}, \lambda)_{\text{adic}} &\rightarrow D(\mathcal{X}, \lambda)_{\text{adic}}; \\ -\underline{\otimes}_{\mathcal{X}} -: D(\mathcal{X}, \lambda)_{\text{adic}} \times D(\mathcal{X}, \lambda)_{\text{adic}} &\rightarrow D(\mathcal{X}, \lambda)_{\text{adic}}, \\ \underline{\mathbf{Hom}}_{\mathcal{X}}: D(\mathcal{X}, \lambda)_{\text{adic}}^{\text{op}} \times D(\mathcal{X}, \lambda)_{\text{adic}} &\rightarrow D(\mathcal{X}, \lambda)_{\text{adic}}. \end{aligned}$$

The pairs  $(\underline{f}^*, \underline{f}_*)$  and  $(-\underline{\otimes}_{\mathcal{X}}, \underline{\mathbf{Hom}}(\mathcal{X}, -))$  for every  $\mathcal{X} \in D(\mathcal{X}, \lambda)_{\text{adic}}$  are pairs of adjoint functors.

We fix a nonempty set  $L$  of rational primes. If  $\mathcal{X}$  is  $L$ -coprime,  $f: \mathcal{Y} \rightarrow \mathcal{X}$  is locally of finite type, and  $\lambda$  is  $L$ -torsion ringed diagram, then we have another pair of adjoint functors:

$$\underline{f}_!: D(\mathcal{Y}, \lambda)_{\text{adic}} \rightarrow D(\mathcal{X}, \lambda)_{\text{adic}}, \quad \underline{f}^!: D(\mathcal{X}, \lambda)_{\text{adic}} \rightarrow D(\mathcal{Y}, \lambda)_{\text{adic}}.$$

These operations satisfy the similar properties as in the non-adic version (see Propositions 1.5.1 and 1.5.2). In particular, we have the following theorem.

**Theorem 0.1.1 (Base Change).** *Let  $\lambda$  be an  $L$ -torsion ringed diagram, and let*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Z} \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

be a Cartesian square of  $\mathbf{L}$ -coprime Artin stacks where  $p$  is locally of finite type. Then we have a natural isomorphism of functors:

$$f^* \circ p_! \simeq q_! \circ g^* : D_{\text{cart}}(\mathcal{Z}, \lambda)_{\text{adic}} \rightarrow D_{\text{cart}}(\mathcal{Y}, \lambda)_{\text{adic}}.$$

The adic formalism introduced above does *not* assume the constructibility at the first place. In other words, we are free to talk about adic complexes for any sheaves. In particular, in terms of Grothendieck’s fonctions-faisceaux dictionary, we make sense of divergent integrals on stacks over finite fields, those appear for example in [8].

In §2, we introduce a special case of the adic formalism, namely, the  $\mathfrak{m}$ -adic formalism on which there is a good notion of constructibility. Such formalism is enough for most applications. Let  $\Lambda$  be a ring and  $\mathfrak{m} \subseteq \Lambda$  be a principal ideal, satisfying the conditions in Definition 2.1.1. The typical example is that  $\Lambda$  is a 1-dimensional valuation ring and  $\mathfrak{m}$  is a proper ideal. The pair  $(\Lambda, \mathfrak{m})$  corresponds to a ringed diagram  $\Lambda_\bullet$  with the underlying category  $\mathbb{N} = \{0 \rightarrow 1 \rightarrow 2 \rightarrow \dots\}$  and  $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$ . We call this setup as the  $\mathfrak{m}$ -adic formalism. Now we fix a pair  $(\Lambda, \mathfrak{m})$  as above such that  $\Lambda$  is Noetherian and  $\Lambda/\mathfrak{m}$  is  $\mathbf{L}$ -torsion. Let  $S$  be either a quasi-excellent finite-dimensional scheme or a regular scheme of dimension  $\leq 1$ , that is  $\mathbf{L}$ -coprime. Consider Artin stacks that are locally of finite type over  $S$ . In this setup, we define the intersection  $D_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}, \Lambda_\bullet) \cap D(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}$  of constructible complexes and adic complexes as the category of constructible adic complexes. We denote this category by  $D_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}$ , which is a full subcategory of  $D(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}$ . In §2.2<sup>1</sup>, we show that the usual  $t$ -structure on  $D(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}$  restricts to the one on  $D_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}$ . Moreover, the six operations mentioned previously restrict to the following refined ones:

$$\begin{aligned} \underline{f}^* : D_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}} &\rightarrow D_{\text{cons}}(\mathcal{Y}, \Lambda_\bullet)_{\text{adic}}, & \underline{f}^! : D_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}} &\rightarrow D_{\text{cons}}(\mathcal{Y}, \Lambda_\bullet)_{\text{adic}}; \\ -\otimes_{\mathcal{X}}^- : D_{\text{cons}}^{(-)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}} \times D_{\text{cons}}^{(-)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}} &\rightarrow D_{\text{cons}}^{(-)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}, \\ \underline{\text{Hom}}_{\mathcal{X}} : D_{\text{cons}}^{(b)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}^{\text{op}} \times D_{\text{cons}}^{(+)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}} &\rightarrow D_{\text{cons}}^{(+)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}. \end{aligned}$$

If  $f$  is *quasi-compact and quasi-separated*, then we have

$$\underline{f}_* : D_{\text{cons}}^{(+)}(\mathcal{Y}, \Lambda_\bullet)_{\text{adic}} \rightarrow D_{\text{cons}}^{(+)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}, \quad \underline{f}_! : D_{\text{cons}}^{(-)}(\mathcal{Y}, \Lambda_\bullet)_{\text{adic}} \rightarrow D_{\text{cons}}^{(-)}(\mathcal{X}, \Lambda_\bullet)_{\text{adic}}.$$

In §2.5, we show that our theory of constructible adic formalism coincides with Laszlo–Olsson [15] under their assumptions.

In §3, we define the perverse  $t$ -structure, in both non-adic and adic setting, for general “perversity” for (higher) Artin stacks, while in all previous theory only middle perversity is considered [15]. We define a *perversity smooth (resp. étale) evaluation*  $\mathfrak{p}$  on an Artin (resp. a Deligne–Mumford) stack  $\mathcal{X}$  (Definition 3.1.8) to be an assignment to each atlas  $u : X \rightarrow \mathcal{X}$  a weak perversity function  $\mathfrak{p}_u$  on  $X$  in the sense of Gabber [9], satisfying certain compatibility condition. In particular, when  $\mathcal{X}$  is a scheme, a perversity étale evaluation is same as a weak perversity function.

**Theorem 0.1.2** ((Adic) perverse  $t$ -structure, §§3.2, 3.3). *Let  $\mathcal{X}$  be an  $\mathbf{L}$ -coprime Artin stack equipped with a perversity smooth evaluation  $\mathfrak{p}$  and  $\lambda$  be an  $\mathbf{L}$ -torsion ringed diagram.*

- (1) *There is a unique up to equivalence  $t$ -structure  $({}^{\mathfrak{p}}D^{\leq 0}(\mathcal{X}, \lambda), {}^{\mathfrak{p}}D^{\geq 0}(\mathcal{X}, \lambda))$  on  $D(\mathcal{X}, \lambda) = D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \lambda)$ , called the perverse  $t$ -structure, such that for every atlas  $u : X \rightarrow \mathcal{X}$ ,  $u^* {}^{\mathfrak{p}}D^{\leq 0}(\mathcal{X}, \lambda) = {}^{\mathfrak{p}}u D^{\leq 0}(X, \lambda)$  and  $u^* {}^{\mathfrak{p}}D^{\geq 0}(\mathcal{X}, \lambda) = {}^{\mathfrak{p}}u D^{\geq 0}(X, \lambda)$ , where the corresponding  $t$ -structure on the scheme  $X$  is defined by Gabber [9].*
- (2) *If  $f : \mathcal{Y} \rightarrow \mathcal{X}$  is a smooth morphism, then  $f^*$  is perverse  $t$ -exact with respect to compatible perversity smooth evaluations  $\mathfrak{p}$  on  $\mathcal{X}$  and  $\mathfrak{q}$  on  $\mathcal{Y}$ .*
- (3) *We have above results in the adic setting, where  ${}^{\mathfrak{p}}D^{\leq 0}(\mathcal{X}, \lambda)_{\text{adic}} = {}^{\mathfrak{p}}D^{\leq 0}(\mathcal{X}, \lambda) \cap D(\mathcal{X}, \lambda)_{\text{adic}}$ .*
- (4) *Moreover, the classical description of the perverse  $t$ -structure via cohomology on stalks again holds (Propositions 3.2.7 and 3.3.2).*

<sup>1</sup>We actually denote the  $(\infty)$ -category of constructible adic complexes by  $\underline{D}_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)$  in §2.2. See §0.2 for an explanation.

In particular, when  $\mathfrak{p} = 0$ , we recover the usual  $t$ -structure in the non-adic case and obtain the similar usual  $t$ -structure in the adic case. When  $\mathfrak{p}$  is the middle perversity evaluation, we generalize the classical notion of middle perverse  $t$ -structure for schemes, in both non-adic and adic cases.

In §3.4, we show that under certain conditions on  $(\Lambda, \mathfrak{m})$  and the perversity smooth evaluation  $\mathfrak{p}$ , the adic perverse  $t$ -structure restricts to the one on  $D_{\text{cons}}(\mathcal{X}, \Lambda_{\bullet})_{\text{adic}}$ . In particular, when  $\mathfrak{p}$  is the middle perversity smooth evaluation (that is, the middle perversity function in the case of schemes), the corresponding (adic) perverse  $t$ -structure coincides with the one defined by Laszlo–Olsson [15], under their further restrictions on  $(\Lambda, \mathfrak{m})$  and  $\mathcal{X}$ .

In §4, we prove several additional  $\infty$ -categorical descent properties of derived  $\infty$ -categories and their adic version we have constructed. In particular, we have the following theorem, which is the incarnation on the level of usual derived categories of the main result in §4.1.

**Theorem 0.1.3** (Proposition 4.1.9). *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be morphism of Artin stacks and  $y: \mathcal{Y}_0^+ \rightarrow \mathcal{Y}$  be a smooth surjective morphism. Let  $\mathcal{Y}_{\bullet}^+$  be a smooth hypercovering of  $\mathcal{Y}$  with the morphism  $y_n: \mathcal{Y}_n^+ \rightarrow \mathcal{Y}_{-1}^+ = \mathcal{Y}$ . Put  $f_n = f \circ y_n: \mathcal{Y}_n^+ \rightarrow \mathcal{X}$ .*

- (1) *For every complex  $\mathcal{K}$  in  $D^{\geq 0}(\mathcal{Y}, \lambda)$  (resp.  $D^{\geq 0}(\mathcal{Y}, \lambda)_{\text{adic}}$ ), we have a convergent spectral sequence*

$$E_1^{p,q} = H^q(f_{p*} y_p^* \mathcal{K}) \Rightarrow H^{p+q} f_* \mathcal{K} \quad (\text{resp. } \underline{E}_1^{p,q} = H^q(\underline{f}_{p*} \underline{y}_p^* \mathcal{K}) \Rightarrow H^{p+q} \underline{f}_* \mathcal{K}).$$

- (2) *If  $\mathcal{X}$  is  $\mathbb{L}$ -coprime;  $\lambda$  is  $\mathbb{L}$ -torsion, and  $f$  is locally of finite type, then for every complex  $\mathcal{K}$  in  $D^{\leq 0}(\mathcal{Y}, \lambda)$  (resp.  $D^{\leq 0}(\mathcal{Y}, \lambda)_{\text{adic}}$ ), we have a convergent spectral sequence*

$$\tilde{E}_1^{p,q} = H^q(f_{-p!} y_{-p}^! \mathcal{K}) \Rightarrow H^{p+q} f_! \mathcal{K} \quad (\text{resp. } \tilde{\underline{E}}_1^{p,q} = H^q(\underline{f}_{-p!} \underline{y}_{-p}^! \mathcal{K}) \Rightarrow H^{p+q} \underline{f}_! \mathcal{K}).$$

Finally, we would like to emphasize that all conventions and notation from [18], especially those in §0.5 there, will be continually adopted in the current article, unless otherwise specified.

**0.2. Refinement of Deligne’s construction.** Let  $\Lambda$  be the ring of integers of a finite extension of  $\mathbb{Q}_{\ell}$ ,  $\mathfrak{m}$  be the maximal ideal of  $\Lambda$ . Let  $X$  be a Noetherian separated algebraic space over  $\mathbb{Z}[1/\ell]$ . Deligne [6, 1.1.2] defines a category  $D^-(X, \Lambda_{\bullet})_{\text{adic}}$  (which he denotes by  $D^-(X, \Lambda)$ ) as the 2-limit of  $D^-(X, \Lambda_n)$ , for  $n \in \mathbb{N}$ , where the transition functors are the derived tensor  $-\otimes_{\Lambda_n}^{\mathbb{L}} \Lambda_m$  for  $m \rightarrow n$ , in the 2-category of categories. Moreover, he defines a subcategory  $D_{\text{cons}}^b(X, \Lambda_{\bullet})$  as the 2-limit of  $D_{\text{cons, ftd}}^b(X, \Lambda_n)$ , where ftd stand for finite Tor-dimension. If  $\text{Hom}(K_n, L_n)$  is finite for all  $K, L \in D_{\text{cons}}^b(X, \Lambda_{\bullet})$  (this is the case if  $X$  is of finite type over a finite field or an algebraically closed field), then  $D_{\text{cons}}^b(X, \Lambda_{\bullet})$  is a triangulated category.

We provide a refinement of Deligne’s construction via limit in (part of) §§1.2, 1.3 and 1.4. Indeed, if we consider the enhancement  $\mathcal{D}(\mathcal{X}, \lambda)_{\text{adic}}$ , which is a presentable stable  $\infty$ -category, of  $D(\mathcal{X}, \lambda)_{\text{adic}}$ , then  $\mathcal{D}(\mathcal{X}, \lambda)_{\text{adic}}$  is naturally the  $\infty$ -categorical limit of the diagram  $\xi \rightarrow \mathcal{D}(\mathcal{X}, \Lambda(\xi))$ , where the transition functors are the ( $\infty$ -categorical) derived tensor  $-\otimes_{\Lambda(\xi')}^{\mathbb{L}} \Lambda(\xi)$  for  $\xi \rightarrow \xi'$ . Here,  $\lambda = (\Xi, \Lambda)$  is an arbitrary ringed diagram and  $\mathcal{X}$  is an Artin stack (see Corollary 1.4.3).

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## 1. THE ADIC FORMALISM

In this chapter, we provide the adic formalism. In §1.1, we work in the general nonsense and define the abstract “adic objects” in a system assigning each coefficient a (diagram of)  $\infty$ -category. We then define the adic category to be the full subcategory spanned by adic objects. In §1.2, we study a natural and fundamental relation between the adic category and certain limit category constructing from the same system, refining a consideration of Deligne. In §1.3, we record some simple but important properties of the system arising from algebraic geometry, namely, those  $\infty$ -categories that are derived categories of (lisse-)étale sheaves on schemes or Artin stacks. In this geometric case, we may identify the adic category and the limit category mentioned previously, which is proved in §1.4. In §1.5, we construct the enhanced adic operation maps and study the usual  $t$ -structure on the adic categories. §1.6, we introduce the adic dualizing complex and prove the biduality property, which will be used later to prove the compatibility between our theory and Laszlo–Olsson’s [13, 14] under their restrictions.

**1.1. Adic objects.** Let  $\mathcal{R} \subseteq \mathcal{R}\text{ing}$  be a full subcategory. We denote by  $\mathcal{R}\text{ind}_{\mathcal{R}} \subseteq \mathcal{R}\text{ind}$  the full subcategory spanned by those ringed diagrams  $(\Xi, \Lambda)$  such that  $\Lambda: \Xi^{op} \rightarrow \mathcal{R}\text{ing}$  factorizes through  $\mathcal{R}$ . In particular, if  $\mathcal{R}$  is the subcategory spanned by torsion (resp.  $L$ -torsion) rings, then  $\mathcal{R}\text{ind}_{\mathcal{R}}$  is  $\mathcal{R}\text{ind}_{\text{tor}}$  (resp.  $\mathcal{R}\text{ind}_{L\text{-tor}}$ ).

Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{P}\text{r}_{\text{st}}^L$  be a functor. We denote by  $\mathcal{C}(\Xi, \Lambda)$  the image of  $(\Xi, \Lambda)$  under  $\mathcal{C}$ . Fix an object  $(\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ . For every morphism  $\varphi: \xi \rightarrow \xi'$  in  $\Xi$ , there is a commutative diagram in  $\mathcal{R}\text{ind}_{\mathcal{R}}$  of the form

$$\begin{array}{ccccc} (\Xi, \Lambda) & \xleftarrow{i_{\xi}} & (\Xi/\xi, \Lambda/\xi) & \xrightarrow{p_{\xi}} & (\{\xi\}, \Lambda(\xi)) \\ \downarrow = & & \downarrow i_{\varphi} & & \downarrow \tilde{\varphi} \\ (\Xi, \Lambda) & \xleftarrow{i_{\xi'}} & (\Xi/\xi', \Lambda/\xi') & \xrightarrow{p_{\xi'}} & (\{\xi'\}, \Lambda(\xi')), \end{array}$$

which induces the following diagram in  $\mathcal{P}\text{r}_{\text{st}}^L$  by applying  $\mathcal{C}$ :

$$\begin{array}{ccccc} \mathcal{C}(\Xi, \Lambda) & \xrightarrow{i_{\xi}^*} & \mathcal{C}(\Xi/\xi, \Lambda/\xi) & \xleftarrow{p_{\xi}^*} & \mathcal{C}(\{\xi\}, \Lambda(\xi)) \\ \uparrow = & & \uparrow i_{\varphi}^* & & \uparrow \tilde{\varphi}^* \\ \mathcal{C}(\Xi, \Lambda) & \xrightarrow{i_{\xi'}^*} & \mathcal{C}(\Xi/\xi', \Lambda/\xi') & \xleftarrow{p_{\xi'}^*} & \mathcal{C}(\{\xi'\}, \Lambda(\xi')), \end{array}$$

where  $-^* = \mathcal{C}(-)$ . Let  $p_{\xi*}$  (resp.  $p_{\xi'*}$ ) be a right adjoint of  $p_{\xi}^*$  (resp.  $p_{\xi'}^*$ ) and  $\alpha_{\varphi}: \tilde{\varphi}^* p_{\xi'*} \rightarrow p_{\xi*} i_{\varphi}^*$  be the natural transformation.

**Definition 1.1.1** (Adic object). An object  $\mathcal{K}$  of  $\mathcal{C}(\Xi, \Lambda)$  is *adic* (with respect to  $\mathcal{C}$ ) if the natural morphism

$$\tilde{\varphi}^* p_{\xi'*} i_{\xi'}^* \mathcal{K} \xrightarrow{\alpha_{\varphi}(i_{\xi'}^* \mathcal{K})} p_{\xi*} i_{\varphi}^* i_{\xi'}^* \mathcal{K}$$

is an equivalence for every morphism  $\varphi: \xi \rightarrow \xi'$  in  $\Xi$ . The target of  $\alpha_{\varphi}(i_{\xi'}^* \mathcal{K})$  is equivalent to  $p_{\xi*} i_{\xi}^* \mathcal{K}$ . It is clear that adic objects are stable under equivalence.

Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{P}\text{r}_{\text{st}}^L$  be a functor and  $(\Xi, \Lambda)$  be an object of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ . We denote by  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \subseteq \mathcal{C}(\Xi, \Lambda)$  the (strictly) full subcategory spanned by adic objects. It is clearly a stable  $\infty$ -category and the inclusion is an exact functor. We emphasize that the full subcategory  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  depends on the functor  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{P}\text{r}_{\text{st}}^L$ , not just on  $\mathcal{C}(\Xi, \Lambda)$ . If  $\mathcal{C}'$  is another such functor, then  $(\mathcal{C} \times \mathcal{C}')(\Xi, \Lambda)_{\text{adic}}$  is equivalent to  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \times \mathcal{C}'(\Xi, \Lambda)_{\text{adic}}$ .

**Definition 1.1.2.**

- (1) A functor  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{C}\text{at}_{\infty}$  is *topological* if
- (a)  $\mathcal{C}$  factorizes through  $\mathcal{P}\text{r}_{\text{st}}^{\text{L}}$ .
  - (b) For every object  $(\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$  such that  $\Xi$  admits a final object,  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  is presentable.
  - (c) For every morphism  $(\Gamma, \gamma): (\Xi^0, \Lambda^0) \rightarrow (\Xi^1, \Lambda^1)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$  that carries the final object  $\xi^0$  of  $\Xi^0$  to the final object  $\xi^1$  of  $\Xi^1$ , the following diagram

$$\begin{array}{ccc} \mathcal{C}(\Xi^1, \Lambda^1) & \xleftarrow{p_{\xi^1}^*} & \mathcal{C}(\{\xi^1\}, \Lambda^1(\xi^1)) \\ \downarrow & & \downarrow \\ \mathcal{C}(\Xi^0, \Lambda^0) & \xleftarrow{p_{\xi^0}^*} & \mathcal{C}(\{\xi^0\}, \Lambda^0(\xi^0)) \end{array}$$

is right adjointable. Moreover, its right adjoint is in  $\mathcal{P}\text{r}_{\text{st}}^{\text{L}}$ .

- (2) A morphism  $\delta: \mathcal{C}^0 \rightarrow \mathcal{C}^1$  in  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  is *topological* if
- (a) Both  $\mathcal{C}^0$  and  $\mathcal{C}^1$  are topological.
  - (b) For every object  $(\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$  with the final object  $\xi$ , the following diagram

$$\begin{array}{ccc} \mathcal{C}^0(\Xi, \Lambda) & \xleftarrow{p_{\xi}^{0*}} & \mathcal{C}^0(\{\xi\}, \Lambda(\xi)) \\ \downarrow & & \downarrow \\ \mathcal{C}^1(\Xi, \Lambda) & \xleftarrow{p_{\xi}^{1*}} & \mathcal{C}^1(\{\xi\}, \Lambda(\xi)) \end{array}$$

is right adjointable.

- (3) We denote by  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  (resp.  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\text{r}_{\text{st}}^{\text{L}})$ ) the subcategory of  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  (resp.  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\text{r}_{\text{st}}^{\text{L}})$ ) spanned by topological functors and topological morphisms. In other words, an  $n$ -cell of  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  (resp.  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\text{r}_{\text{st}}^{\text{L}})$ ) belongs to  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  (resp.  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\text{r}_{\text{st}}^{\text{L}})$ ) if all its vertices are topological functors and edges are topological morphisms.

**Lemma 1.1.3.** *We have*

- (1) Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{C}\text{at}_{\infty}$  be a topological functor. Then for every morphism  $(\Gamma, \gamma): (\Xi^0, \Lambda^0) \rightarrow (\Xi^1, \Lambda^1)$  in  $\mathcal{R}\text{ind}_{\mathcal{R}}$ , the functor  $\mathcal{C}(\Gamma, \gamma): \mathcal{C}(\Xi^1, \Lambda^1) \rightarrow \mathcal{C}(\Xi^0, \Lambda^0)$  preserves adic objects, that is, it carries adic objects to adic objects.
- (2) Let  $\delta: \mathcal{C}^0 \rightarrow \mathcal{C}^1$  be a topological morphism. Then for every object  $(\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ , the functor  $\delta(\Xi, \Lambda): \mathcal{C}^0(\Xi, \Lambda) \rightarrow \mathcal{C}^1(\Xi, \Lambda)$  preserves adic objects.

*Proof.* We prove (2), and (1) is similar. Let  $\varphi: \xi \rightarrow \xi'$  be a morphism in  $\Xi$ . Consider the following diagram with all solid squares commutative (up to obvious homotopies):

$$\begin{array}{ccccc}
 & \mathcal{C}^0(\Xi, \Lambda) & \xrightarrow{i_\xi^{0*}} & \mathcal{C}^0_{/\xi} & \xrightarrow{p_{\xi*}^0} & \mathcal{C}^0(\xi) \\
 & \delta(\Xi, \Lambda) \swarrow & & \delta_{/\xi} \swarrow & & \delta(\xi) \swarrow \\
 \mathcal{C}^1(\Xi, \Lambda) & \xrightarrow{i_\xi^{1*}} & \mathcal{C}^1_{/\xi} & \xrightarrow{p_{\xi*}^1} & \mathcal{C}^1(\xi) & \\
 & \uparrow = & \uparrow & \uparrow i_\varphi^{0*} & & \uparrow \tilde{\varphi}^{0*} \\
 & \mathcal{C}^0(\Xi, \Lambda) & \xrightarrow{i_{\xi'}^{0*}} & \mathcal{C}^0_{/\xi'} & \xrightarrow{p_{\xi'*}^0} & \mathcal{C}^0(\xi') \\
 & \delta(\Xi, \Lambda) \swarrow & & \delta_{/\xi'} \swarrow & & \delta(\xi') \swarrow \\
 \mathcal{C}^1(\Xi, \Lambda) & \xrightarrow{i_{\xi'}^{1*}} & \mathcal{C}^1_{/\xi'} & \xrightarrow{p_{\xi'*}^1} & \mathcal{C}^1(\xi') & \\
 & \uparrow = & \uparrow & \uparrow i_\varphi^{1*} & & \uparrow \tilde{\varphi}^{1*}
 \end{array}$$

where we have written  $\delta_{/\xi}: \mathcal{C}^0_{/\xi} \rightarrow \mathcal{C}^1_{/\xi}$  instead of  $\delta(\Xi/\xi, \Lambda/\xi): \mathcal{C}^0(\Xi/\xi, \Lambda/\xi) \rightarrow \mathcal{C}^1(\Xi/\xi, \Lambda/\xi)$  and  $\delta(\xi): \mathcal{C}^0(\xi) \rightarrow \mathcal{C}^1(\xi)$  instead of  $\delta(\{\xi\}, \Lambda(\xi)): \mathcal{C}^0(\{\xi\}, \Lambda(\xi)) \rightarrow \mathcal{C}^1(\{\xi\}, \Lambda(\xi))$  for short. By definition, we need to show that for every adic object  $\mathcal{K}$  of  $\mathcal{C}^0(\Xi, \Lambda)$ , the natural morphism

$$\tilde{\varphi}^{1*} p_{\xi'*}^1 i_{\xi'}^{1*} \delta(\Xi, \Lambda) \mathcal{K} \xrightarrow{\alpha_\varphi^1(i_{\xi'}^{1*} \delta(\Xi, \Lambda) \mathcal{K})} p_{\xi*}^1 i_\xi^{1*} \delta(\Xi, \Lambda) \mathcal{K}$$

is an equivalence. But we have the following commutative diagram in  $\mathbf{h}\mathcal{C}^1(\xi)$

$$\begin{array}{ccc}
 \delta(\xi) \tilde{\varphi}^{0*} p_{\xi'*}^0 i_{\xi'}^{0*} \mathcal{K} \simeq \tilde{\varphi}^{1*} \delta(\xi') p_{\xi'*}^0 i_{\xi'}^{0*} \mathcal{K} & \longrightarrow & \tilde{\varphi}^{1*} p_{\xi'*}^1 \delta_{/\xi'} i_{\xi'}^{0*} \mathcal{K} \simeq \tilde{\varphi}^{1*} p_{\xi'*}^1 i_{\xi'}^{1*} \delta(\Xi, \Lambda) \mathcal{K} \\
 \downarrow \delta(\xi)(\alpha_\varphi^0(i_{\xi'}^{0*} \mathcal{K})) & & \downarrow \alpha_\varphi^1(i_{\xi'}^{1*} \delta(\Xi, \Lambda) \mathcal{K}) \\
 \delta(\xi) p_{\xi*}^0 i_\phi^{0*} i_{\xi'}^{0*} \mathcal{K} \simeq \delta(\xi) p_{\xi*}^0 i_\xi^{0*} \mathcal{K} & \longrightarrow & p_{\xi*}^1 \delta_{/\xi} i_\xi^{0*} \mathcal{K} \simeq p_{\xi*}^1 i_\xi^{1*} \delta(\Xi, \Lambda) \mathcal{K},
 \end{array}$$

where the horizontal arrows are equivalences since  $\delta$  is topological. Therefore,  $\alpha_\varphi^1(i_{\xi'}^{1*} \delta(\Xi, \Lambda) \mathcal{K})$  is an equivalence since  $\alpha_\varphi^0(i_{\xi'}^{0*} \mathcal{K})$  is.  $\square$

**Lemma 1.1.4.** *The  $\infty$ -category  $\mathrm{Fun}^{\mathrm{top}}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}})$  admits small limits and those limits are preserved under the inclusion  $\mathrm{Fun}^{\mathrm{top}}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}}) \subseteq \mathrm{Fun}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}})$ .*

*Proof.* We will frequently use [20, 6.2.3.18] in this article.

First, we show that for a limit diagram  $\mathcal{C}: K^\triangleleft \rightarrow \mathrm{Fun}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}})$  such that  $\mathcal{C}|K$  factorizes through  $\mathrm{Fun}^{\mathrm{top}}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}})$ , then the limit functor  $\mathcal{C}^{-\infty}$  is topological. To check the requirement (1c), we pick a morphism  $(\Gamma, \gamma): (\Xi^0, \Lambda^0) \rightarrow (\Xi^1, \Lambda^1)$  of  $\mathcal{R}\mathrm{ind}_{\mathcal{R}}$  that sends the final object  $\xi^0$  of  $\Xi^0$  to the final object  $\xi^1$  of  $\Xi^1$ . By assumption, the functor  $(\mathcal{C}|K)|(\Delta^1 \times \Delta^1)^{\mathrm{op}}$ , where  $\Delta^1 \times \Delta^1$  is the diagram

$$\begin{array}{ccc}
 (\Xi^0, \Lambda^0) & \xrightarrow{p_{\xi^0}} & (\{\xi^0\}, \Lambda^0(\xi^0)) \\
 (\Gamma, \gamma) \downarrow & & \downarrow \\
 (\Xi^1, \Lambda^1) & \xrightarrow{p_{\xi^1}} & (\{\xi^1\}, \Lambda^1(\xi^1)),
 \end{array}$$

can be viewed as a functor  $K \times \Delta^1 \rightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat}_\infty)$ . By [20, 6.2.3.18] and the fact that  $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}} \subseteq \mathrm{Cat}_\infty$  preserves small limits, the limit functor  $\mathcal{C}^{-\infty}$  satisfies (c), and the limit diagram factorizes through  $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}}$ . In particular, for every object  $k$  of  $K$ , the natural morphism  $\mathcal{C}^{-\infty} \rightarrow \mathcal{C}^k = \mathcal{C}(k)$  satisfies condition (2b). To check that  $\mathcal{C}^{-\infty}$  satisfies condition (1b), we only need to show that the diagram  $\mathcal{D}: K^\triangleleft \rightarrow \mathrm{Fun}(\mathbf{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{\mathrm{op}}, \mathrm{Cat}_\infty)$  that carries  $k$  to  $(\Xi, \Lambda) \mapsto \mathcal{C}^k(\Xi, \Lambda)_{\mathrm{adic}}$  is a limit



diagram. Let  $\mathcal{C}^\infty(\Xi, \Lambda)'$  be the limit  $\varprojlim_K \mathcal{C}^k(\Xi, \Lambda)_{\text{adic}}$ , which is naturally a strictly full subcategory of  $\mathcal{C}^\infty(\Xi, \Lambda)$  and contains  $\mathcal{C}^\infty(\Xi, \Lambda)_{\text{adic}}$  by [18, 3.1.5] and the proof of Lemma 1.1.3 (2). We need to prove  $\mathcal{C}^\infty(\Xi, \Lambda)' \supseteq \mathcal{C}^\infty(\Xi, \Lambda)_{\text{adic}}$ . By definition, we may assume  $\Xi$  admits a final object  $\xi'$  and need to show that for every other object  $\xi \in \Xi$ , the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\Xi/\xi, \Lambda/\xi)' & \xleftarrow{p_\xi^*} & \mathcal{C}^\infty(\{\xi\}, \Lambda(\xi)) \\ i_\varphi^* \uparrow & & \uparrow \bar{\varphi}^* \\ \mathcal{C}^\infty(\Xi, \Lambda)' & \xleftarrow{p_{\xi'}^*} & \mathcal{C}^\infty(\{\xi'\}, \Lambda(\xi')) \end{array}$$

induced by the map  $\phi: \xi \rightarrow \xi'$  is right adjointable. This follows from the fact that, for every  $k \in K$ , the corresponding diagram with  $\mathcal{C}^\infty(-)$  replaced by  $\mathcal{C}^k(-)$  and  $\mathcal{C}^\infty(-)'$  replaced by  $\mathcal{C}^k(-)_{\text{adic}}$  is right adjointable.

Second, we show that for an arbitrary diagram  $\mathcal{C}: K^\triangleleft \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{\text{op}}, \mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}})$ , the induced morphism  $\mathcal{C}^{-\infty} \rightarrow \varprojlim(\mathcal{C} \mid K)$  in  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{\text{op}}, \text{Cat}_\infty)$  is topological. To check condition (2b), we pick an object  $(\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$  with a final object  $\xi$  of  $\Xi$ . Then this again follows from [20, 6.2.3.18].  $\square$

**Lemma 1.1.5.** *Let  $\mathcal{D}$  be a full subcategory of an  $\infty$ -category  $\mathcal{C}$  and  $f: \mathcal{D} \rightarrow \mathcal{C}$  be the inclusion. Then the pullback of  $f$  in the category  $\text{Set}_\Delta$  by any functor  $g: \mathcal{C}' \rightarrow \mathcal{C}$  with source in  $\text{Cat}_\infty$  is a pullback in  $\text{Cat}_\infty$ .*

*Proof.* This follows immediately from [18, 3.1.5] applied to the pullback of  $\text{id}_{\mathcal{C}}$  by  $g$ .  $\square$

**Proposition 1.1.6.** *Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{\text{op}} \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$  be a topological functor. For every object  $(\Xi, \Lambda)$ , the inclusion  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \rightarrow \mathcal{C}(\Xi, \Lambda)$  is a morphism in  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$ .*

By the proposition, the inclusion  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \rightarrow \mathcal{C}(\Xi, \Lambda)$  admits a right adjoint  $R: \mathcal{C}(\Xi, \Lambda) \rightarrow \mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  which we call the *colocalization functor*.

*Proof.* By definition, the inclusion  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \subseteq \mathcal{C}(\Xi, \Lambda)$  fits into the following diagram

$$\begin{array}{ccc} \mathcal{C}(\Xi, \Lambda)_{\text{adic}} & \xrightarrow{\quad} & \prod_{\xi \in \text{Ob}(\Lambda)} \mathcal{C}(\Xi/\xi, \Lambda/\xi)_{\text{adic}} \\ \downarrow & & \downarrow \\ \mathcal{C}(\Xi, \Lambda) & \xrightarrow{\prod_{\xi \in \text{Ob}(\Lambda)} i_\xi^*} & \prod_{\xi \in \text{Ob}(\Lambda)} \mathcal{C}(\Xi/\xi, \Lambda/\xi), \end{array}$$

which is a pullback diagram in  $\text{Cat}_\infty$  by the above lemma. Since  $p_{\xi*}$  commutes with small colimits by condition (1c) of Definition 1.1.2,  $\mathcal{C}(\Xi/\xi, \Lambda/\xi)_{\text{adic}}$  admits small colimits and the inclusion into  $\mathcal{C}(\Xi/\xi, \Lambda/\xi)$  preserves such colimits. By condition (1b) of Definition 1.1.2, the source is presentable (and stable), so the above inclusion is a morphism in  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$ . Therefore, the right vertical arrow is a morphism in  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$  since  $\Lambda$  is small. Moreover, the functor  $\prod_{\xi \in \text{Ob}(\Lambda)} i_\xi^*$  preserves small colimits since each  $i_\xi^*$  does and  $\Lambda$  is small. Therefore, the inclusion  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \rightarrow \mathcal{C}(\Xi, \Lambda)$  is a morphism in  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$ , because the inclusion  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}} \subseteq \text{Cat}_\infty$  preserves small limits.  $\square$

**Corollary 1.1.7.** *Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{\text{op}} \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\text{L}}$  be a topological functor and  $(\Xi, \Lambda)$  be an object of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ . Let  $F: \mathcal{C}(\Xi, \Lambda) \rightarrow \mathcal{C}(\Xi, \Lambda)$  be a functor admitting right adjoints such that the set  $F(\mathcal{C}(\Xi, \Lambda)_{\text{adic}})$  is contained in  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$ . Then the induced functor  $F|_{\mathcal{C}(\Xi, \Lambda)_{\text{adic}}}: \mathcal{C}(\Xi, \Lambda)_{\text{adic}} \rightarrow \mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  admits right adjoints as well.*

*Proof.* By the above proposition,  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  is presentable (without assuming that  $\Xi$  admits the final object). Therefore, by Adjoint Functor Theorem, we only need to show that  $F|_{\mathcal{C}(\Xi, \Lambda)_{\text{adic}}}$  preserves small colimits. This then follows from the fact that the inclusion  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \subseteq \mathcal{C}(\Xi, \Lambda)$  preserves small colimits, and the assumptions on  $F$ .  $\square$



To conclude this section, we introduce another formalism, which makes use of a limit construction instead of adic objects. This can be seen as a refinement of Deligne's construction [6, 1.1.2].

*Notation 1.1.8.* Let  $\mathcal{C}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}$  be a topological functor. For every object  $\lambda = (\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ , there is a tautological functor  $\lambda^*: \Xi \rightarrow \mathcal{R}\text{ind}_{\mathcal{R}}$  sending  $\xi$  to  $(*, \Lambda(\xi))$ . Composing with  $\mathcal{C}$ , we obtain a diagram  $\mathcal{C} \circ \mathcal{N}(\lambda^*)^{op}: \mathcal{N}(\Xi)^{op} \rightarrow \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}$ . Let  $\underline{\mathcal{C}}(\Xi, \Lambda)$  be the limit  $\varprojlim \mathcal{C} \circ \mathcal{N}(\lambda^*)^{op}$ , viewed as an object of  $\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}$ . We will construct a natural functor  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}} \rightarrow \underline{\mathcal{C}}(\Xi, \Lambda)$  in §1.2 (in a coherent way) and show in §1.4 that it is an equivalence if  $\mathcal{C}$  comes from the derived  $\infty$ -categories of (higher) stacks.

## 1.2. The natural transformations $\alpha_{\mathcal{R}}$ and $\alpha_{\mathcal{R}}^{\otimes}$ .

*Notation 1.2.1.* We denote by

$$\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})) \subseteq \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty}))$$

the subcategory spanned by cells that factorize through  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Cat}_{\infty})$ . Here we view  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty}))$  as a subcategory of  $\text{Fun}(\mathcal{P}\mathcal{f}^{\otimes}, \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Cat}_{\infty}))$ .

We construct two endofunctors

$$(1.1) \quad \mathcal{A}\text{dic}_{\mathcal{R}}, \underline{\mathcal{A}}\text{dic}_{\mathcal{R}}: \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})) \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})),$$

and a natural transformation  $\alpha_{\mathcal{R}}: \mathcal{A}\text{dic}_{\mathcal{R}} \rightarrow \underline{\mathcal{A}}\text{dic}_{\mathcal{R}}$ . If we represent objects of the functor category by their compositions with the restriction functor  $G_{\zeta}$  in [18, 1.5.8 (4)], then  $\mathcal{A}\text{dic}_{\mathcal{R}}$  (resp.  $\underline{\mathcal{A}}\text{dic}_{\mathcal{R}}$ ) sends an object  $\mathcal{C}' \times \mathcal{C} \xrightarrow{- \otimes_{\mathcal{C}'} f^{*-}} \mathcal{C}'$  to

$$\mathcal{C}'_{\text{adic}} \times \mathcal{C}_{\text{adic}} \xrightarrow{- \otimes_{\mathcal{C}'} f^{*-}} \mathcal{C}'_{\text{adic}} \quad (\text{resp. } \underline{\mathcal{C}}' \times \underline{\mathcal{C}} \xrightarrow{- \otimes_{\mathcal{C}'} f^{*-}} \underline{\mathcal{C}}').$$

Here,  $\mathcal{C}_{\text{adic}}$  (resp.  $\underline{\mathcal{C}}$ ) takes the value  $\mathcal{C}_{\text{adic}}(\Xi, \Lambda)$  (resp.  $\underline{\mathcal{C}}(\Xi, \Lambda)$ ) at  $(\Xi, \Lambda)$ , which justifies the notation.

By definition, there is a tautological functor  $\mathcal{R}\text{ind}_{\mathcal{R}} \rightarrow \text{Cat}_1$  sending  $(\Xi, \Lambda)$  to  $\Xi$ . Applying Grothendieck's construction, we obtain an op-fibration  $\pi: \mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}} \rightarrow \mathcal{R}\text{ind}_{\mathcal{R}}$ . The objects of  $\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}}$  are pairs  $((\Xi, \Lambda), \xi)$  where  $(\Xi, \Lambda)$  is an object of  $\mathcal{R}\text{ind}_{\mathcal{R}}$  and  $\xi$  is an object of  $\Xi$ . Moreover, there are functors  $p^{\text{univ}}: \mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}} \times [1] \rightarrow \mathcal{R}\text{ind}_{\mathcal{R}}$  sending  $((\Xi, \Lambda), \xi) \times [1]$  to  $p_{\xi}: (\Xi/\xi, \Lambda/\xi) \rightarrow (\{\xi\}, \Lambda(\xi))$ , and  $i^{\text{univ}}: \mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}} \diamond_{\mathcal{R}\text{ind}_{\mathcal{R}}} \mathcal{R}\text{ind}_{\mathcal{R}} \rightarrow \mathcal{R}\text{ind}_{\mathcal{R}}$  sending the morphism  $((\Xi, \Lambda), \xi) \rightarrow (\Xi, \Lambda)$  to  $i_{\xi}: (\Xi/\xi, \Lambda/\xi) \rightarrow (\Xi, \Lambda)$ .

Let

$$\mathcal{C}^{\text{univ}}: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \rightarrow \text{Fun}(\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}, \text{Cat}_{\infty})$$

be the universal functor, and let  $\mathcal{C}_{\text{adic}}^{\text{univ}}$  be the functor with the same source and target as  $\mathcal{C}^{\text{univ}}$  and carrying  $(\Xi, \Lambda)$  to  $(\mathcal{C}, X) \mapsto (G_X \circ \mathcal{C})_{\text{adic}}(\Xi, \Lambda)$ . Here  $G_X: \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty}) \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}$  is the evaluation functor at  $X$  defined in [18, 1.5.8 (4)]. By the definition of adic objects, we can apply partial right adjunction to  $\mathcal{C}_{\text{adic}}^{\text{univ}} \circ \mathcal{N}(p^{\text{univ}})^{op}$  with respect to direction 2 to obtain a functor

$$\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \times \Delta^1 \rightarrow \text{Fun}(\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}, \text{Cat}_{\infty}).$$

Composing this with  $\mathcal{C}_{\text{adic}}^{\text{univ}} \circ \mathcal{N}(i^{\text{univ}})^{op}$ , we obtain a functor

$$F: \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \diamond_{\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}} \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \rightarrow \text{Fun}(\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}}(\text{Cat}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}, \text{Cat}_{\infty}),$$

whose restriction to  $\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}$  is (equivalent to)  $\mathcal{C}_{\text{adic}}^{\text{univ}}$  and restriction to  $\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op}$  is (equivalent to)  $\mathcal{C}_{\text{adic}}^{\text{univ}} \circ \mathcal{N}(t)^{op}$ , where  $t: \mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}} \rightarrow \mathcal{R}\text{ind}_{\mathcal{R}}$  is the functor carrying  $((\Xi, \Lambda), \xi)$  to  $(\{\xi\}, \Lambda(\xi))$ . Let  $F_0$  be the right Kan extension of  $F|_{\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op}}$  along the inclusion  $\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \subseteq \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op} \diamond_{\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}} \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op}$ . Then the restriction of  $F \rightarrow F_0$  to  $\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}$  provides the desired natural transformation  $\alpha_{\mathcal{R}}: \mathcal{A}\text{dic}_{\mathcal{R}} \rightarrow \underline{\mathcal{A}}\text{dic}_{\mathcal{R}}$ . To see this, the only nontrivial point is

the following: By construction and the definition of right Kan extensions, for every object  $\lambda$  of  $\mathcal{R}\text{ind}_{\mathcal{R}}$ ,  $F_0(\lambda)$  is the limit of the diagram

$$\begin{aligned} & \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})_{\lambda/}^{op} \times_{\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}} \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \rightarrow \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \\ & \xrightarrow{F} \text{Fun}(\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}, \mathcal{C}\text{at}_{\infty}), \end{aligned}$$

which is equivalent to the limit of the diagram

$$\begin{aligned} & \{\lambda\} \times_{\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}} \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \rightarrow \mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}}^{\text{univ}})^{op} \\ & \xrightarrow{F} \text{Fun}(\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}, \mathcal{C}\text{at}_{\infty}), \end{aligned}$$

by [19, 4.1.2.10, 4.1.2.15]. The latter diagram, restricted to each fixed object  $(\mathcal{C}, X)$  of  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \times \mathcal{P}\mathcal{f}^{\otimes}$ , is nothing but  $G_X \circ \mathcal{C} \circ \mathcal{N}(\lambda^*)^{op}$ .

*Notation 1.2.2.* We denote by  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})$  the subcategory of  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})$  spanned by cells that factorize through  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty})$  after viewing  $\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})$  as a subcategory of  $\text{Fun}(\mathcal{N}(\mathcal{F}\text{in}_*), \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{C}\text{at}_{\infty}))$ .

We construct two endofunctors

$$(1.2) \quad \mathcal{A}\text{dic}_{\mathcal{R}}^{\otimes}, \underline{\mathcal{A}\text{dic}}_{\mathcal{R}}^{\otimes}: \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes}) \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes}),$$

and a natural transformation  $\alpha_{\mathcal{R}}^{\otimes}: \mathcal{A}\text{dic}_{\mathcal{R}}^{\otimes} \rightarrow \underline{\mathcal{A}\text{dic}}_{\mathcal{R}}^{\otimes}$ , such that  $\mathcal{A}\text{dic}_{\mathcal{R}}^{\otimes}$  (resp.  $\underline{\mathcal{A}\text{dic}}_{\mathcal{R}}^{\otimes}$ ) sends an object  $\mathcal{C}^{\otimes}$  to  $\mathcal{C}_{\text{adic}}^{\otimes}$  (resp.  $\underline{\mathcal{C}}^{\otimes}$ ), where  $(G \circ \mathcal{C}_{\text{adic}}^{\otimes})(\Xi, \Lambda) = (G \circ \mathcal{C}^{\otimes})(\Xi, \Lambda)_{\text{adic}}$  (resp.  $(G \circ \underline{\mathcal{C}}^{\otimes})(\Xi, \Lambda) = (G \circ \mathcal{C}^{\otimes})(\Xi, \Lambda)$ ). Here  $G: \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes} \rightarrow \mathcal{P}\mathcal{r}_{\text{st}}^{\mathcal{L}}$  is the forgetful functor defined in [18, 1.5.3]. The construction of  $\alpha_{\mathcal{R}}^{\otimes}$  is exactly the same as that of  $\alpha_{\mathcal{R}}: \mathcal{A}\text{dic}_{\mathcal{R}} \rightarrow \underline{\mathcal{A}\text{dic}}_{\mathcal{R}}$ .

*Remark 1.2.3.* We have the following properties of compatibility.

(1) If  $\mathcal{R}' \subseteq \mathcal{R}$ , then the following diagram

$$\begin{array}{ccc} \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) & \longrightarrow & \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}'}^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \\ \mathcal{A}\text{dic}_{\mathcal{R}} \downarrow & & \downarrow \mathcal{A}\text{dic}_{\mathcal{R}'} \\ \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) & \longrightarrow & \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}'}^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \end{array}$$

commutes up to homotopy. The same holds for  $\underline{\mathcal{A}\text{dic}}_{\mathcal{R}}$  and  $\alpha_{\mathcal{R}}$ .

(2) The map  $\text{pf}$  [18, (1.1)] induces a map

$$\begin{aligned} & \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{pf}): \text{Fun}((\Delta^1)^{op}, \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})) \\ & \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})). \end{aligned}$$

Moreover, by construction, the following diagram

$$\begin{array}{ccc} \text{Fun}((\Delta^1)^{op}, \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})) & \xrightarrow{\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{pf})} & \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \\ \text{Fun}((\Delta^1)^{op}, \mathcal{A}\text{dic}_{\mathcal{R}}^{\otimes}) \downarrow & & \downarrow \mathcal{A}\text{dic}_{\mathcal{R}} \\ \text{Fun}((\Delta^1)^{op}, \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \mathcal{P}\mathcal{r}_{\text{st,cl}}^{\mathcal{L}\otimes})) & \xrightarrow{\text{Fun}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{pf})} & \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathcal{R}})^{op}, \text{Mon}_{\mathcal{P}\mathcal{f}}^{\mathcal{P}\mathcal{r}\mathcal{L}}(\mathcal{C}\text{at}_{\infty})) \end{array}$$

commutes up to homotopy. We have the same phenomena for  $\underline{\mathcal{A}\text{dic}}_{\mathcal{R}}$  and  $\alpha_{\mathcal{R}}$ .

The following lemma is similar to Lemma 1.1.4 and will be used in §1.3.

**Lemma 1.2.4.** *We have*

- (1) The  $\infty$ -category  $\mathrm{Fun}^{\mathrm{top}}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathrm{Mon}_{\mathrm{pf}}^{\mathcal{P}_{\mathrm{st}}^{\mathrm{L}}}(\mathrm{Cat}_{\infty}))$  admits small limits and those limits are preserved under the inclusion

$$\mathrm{Fun}^{\mathrm{top}}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathrm{Mon}_{\mathrm{pf}}^{\mathcal{P}_{\mathrm{st}}^{\mathrm{L}}}(\mathrm{Cat}_{\infty})) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathrm{Mon}_{\mathrm{pf}}^{\mathcal{P}_{\mathrm{st}}^{\mathrm{L}}}(\mathrm{Cat}_{\infty})).$$

- (2) The  $\infty$ -category  $\mathrm{Fun}^{\mathrm{top}}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L} \otimes})$  admits small limits and those limits are preserved under the inclusion  $\mathrm{Fun}^{\mathrm{top}}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L} \otimes}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathcal{R}})^{op}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L} \otimes})$ .

*Proof.* Both follow from Lemma 1.1.4, (the dual of) [19, 5.1.2.3], and [20, 3.2.2.5].  $\square$

**1.3. Evaluation functors.** Recall that we have maps

$$\begin{aligned} \mathrm{chp}_{\mathrm{L}}^{\mathrm{Ar}} \mathrm{EO} : \delta_{2, \{2\}}^* \mathrm{Fun}(\Delta^1, \mathrm{Chp}_{\mathrm{F}^0, A}^{\mathrm{Ar}})^{\mathrm{cart}} &\rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{L-tor}})^{op}, \mathrm{Mon}_{\mathrm{pf}}^{\mathcal{P}_{\mathrm{st}}^{\mathrm{L}}}(\mathrm{Cat}_{\infty})); \\ \mathrm{chp}^{\mathrm{DM}} \mathrm{EO} : \delta_{2, \{2\}}^* \mathrm{Fun}(\Delta^1, \mathrm{Chp}_{\mathrm{F}^0, A}^{\mathrm{DM}})^{\mathrm{cart}} &\rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{op}, \mathrm{Mon}_{\mathrm{pf}}^{\mathcal{P}_{\mathrm{st}}^{\mathrm{L}}}(\mathrm{Cat}_{\infty})); \\ \mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^* : (\mathrm{Chp}^{\mathrm{Ar}})^{op} &\rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind})^{op}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L} \otimes}). \end{aligned}$$

For an object  $(\Xi, \Lambda)$  of  $\mathcal{R}\mathrm{ind}$  admitting the final object  $\xi \in \Xi$ , we have both the projection  $(\Xi, \Lambda) \xrightarrow{p_{\xi}} (\{\xi\}, \Lambda(\xi))$  and the inclusion  $(\{\xi\}, \Lambda(\xi)) \xrightarrow{s_{\xi}} (\Xi, \Lambda)$ . Now if  $\mathcal{C} = \mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*(X)$ , where  $X$  is a higher Artin stack, then the induced functors  $p_{\xi*}, s_{\xi}^* : \mathcal{C}(\Xi, \Lambda) \rightarrow \mathcal{C}(\{\xi\}, \Lambda(\xi))$  are canonically and coherently equivalent in a sense that we now explain.

Let  $\mathcal{R}\mathrm{ind}^{\mathrm{fin}} \subseteq \mathcal{R}\mathrm{ind}$  be the subcategory spanned by objects  $(\Xi, \Lambda)$  admitting final objects and morphisms preserving final objects. We have natural functors  $\mathcal{R}\mathrm{ind}^{\mathrm{fin}} \rightarrow \mathrm{Fun}([1], \mathcal{R}\mathrm{ind})$  sending  $(\Xi, \Lambda)$  to  $(\Xi, \Lambda) \xrightarrow{p_{\xi}} (\{\xi\}, \Lambda(\xi))$  and  $(\{\xi\}, \Lambda(\xi)) \xrightarrow{s_{\xi}} (\Xi, \Lambda)$ , respectively, where  $\xi$  is the final object of  $\Xi$ .

Apply the functor  $\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  and the forgetful functor  $\mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L} \otimes} \rightarrow \mathrm{Cat}_{\infty}^{\otimes}$ . On the one hand, we obtain a functor

$$\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^* : (\mathrm{Chp}^{\mathrm{Ar}})^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{\mathrm{fin}})^{op} \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}^{\otimes})$$

sending  $(X, (\Xi, \Lambda))$  to  $\mathcal{D}(X, (\{\xi\}, \Lambda(\xi))) \xrightarrow{p_{\xi}^*} \mathcal{D}(X, (\Xi, \Lambda))$ . On the other hand, we have

$$\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^* : (\mathrm{Chp}^{\mathrm{Ar}})^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{\mathrm{fin}})^{op} \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}^{\otimes})$$

sending  $(X, (\Xi, \Lambda))$  to  $\mathcal{D}(X, (\Xi, \Lambda)) \xrightarrow{s_{\xi}^*} \mathcal{D}(X, (\{\xi\}, \Lambda(\xi)))$ . We view these functors as

$$(\mathrm{Chp}^{\mathrm{Ar}})^{op} \times \mathrm{N}(\mathcal{R}\mathrm{ind}^{\mathrm{fin}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}).$$

**Lemma 1.3.1.** *The functor  $\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  factorizes through  $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat}_{\infty})$  and  $\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  factorizes through  $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty})$ . Moreover,  $\mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  is equivalent to  $\epsilon \circ \mathrm{chp}^{\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$ , where  $\epsilon$  is the equivalence  $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty}) \rightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat}_{\infty})$  in [20, 6.2.3.18 (2)].*

In particular,  $p_{\xi}^*$  is a left adjoint of  $s_{\xi}^*$ .

*Proof.* When  $k = -2$ , we have an equivalence between the two functors  $(\mathrm{Sch}^{\mathrm{qc}, \mathrm{sep}})^{op} \times (\mathcal{R}\mathrm{ind})^{op} \rightarrow \mathrm{Ab}$  sending  $(X, (\Xi, \Lambda))$  to *exact and monoidal* functors  $p_{\xi*}, s_{\xi}^* : \mathrm{Mod}(X, (\Xi, \Lambda)) \rightarrow \mathrm{Mod}(X, \Lambda(\xi))$ , respectively. In this case the lemma follows from the construction in [18, 2.2].

In general, we first show by induction on  $k$  that the functor  $\mathrm{chp}^{k-\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  factorizes through  $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat}_{\infty})$ . This holds by descent. Next we prove by induction on  $k$  that  $\mathrm{chp}^{k-\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  and  $\epsilon \circ \mathrm{chp}^{k-\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  are equivalent. This holds by [18, 4.1.1] since smooth surjective morphisms are of both  $\mathrm{chp}^{k-\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$  and  $\mathrm{chp}^{k-\mathrm{Ar}} \mathrm{EO}_{\otimes}^*$ -descent.  $\square$

*Remark 1.3.2.* In general, if  $(\Xi, \Lambda)$  is an object of  $\mathcal{R}\text{ind}$  and  $\xi \in \Xi$ , then we have successive inclusions  $e_\xi: (\{\xi\}, \Lambda(\xi)) \xrightarrow{s_\xi} (\Xi_{/\xi}, \Lambda_{/\xi}) \xrightarrow{i_\xi} (\Xi, \Lambda)$  which induce the *evaluation functor (at  $\xi$ )*

$$e_\xi^*: \mathcal{D}(X, (\Lambda, \Xi)) \rightarrow \mathcal{D}(X, \Lambda(\xi))$$

for a higher Artin stack  $X$ . By the above lemma,  $e_\xi^*$  and  $p_{\xi*} \circ i_\xi^*$  are canonically and coherently equivalent. For brevity, we sometimes also write  $\mathcal{K}_\xi$  for  $e_\xi^* \mathcal{K}$  for an object  $\mathcal{K} \in \mathcal{D}(X, (\Xi, \Lambda))$ .

The functor

$$\prod_{\xi \in \Xi} e_\xi^*: \mathcal{D}(X, (\Lambda, \Xi)) \rightarrow \prod_{\xi \in \Xi} \mathcal{D}(X, \Lambda(\xi))$$

is conservative. This is obvious when  $X$  is in  $\text{Sch}^{\text{qc.sep}}$ . The general case follows, because simplicial limits of conservative functors are conservative.

**Lemma 1.3.3.** *Let  $X$  be a scheme in  $\text{Sch}^{\text{qc.sep}}$ ,  $\mathcal{C} = {}_{\text{chp}^{\text{Ar}}} \text{EO}^*(X): \mathcal{N}(\mathcal{R}\text{ind})^{op} \rightarrow \text{Pr}_{\text{st}}^{\text{L}}$ . Let  $(\Xi, \Lambda)$  be an object of  $\mathcal{R}\text{ind}$  such that  $\Xi$  admits a final object  $\xi$ . Then the image of the natural map  $p_\xi^*: \mathcal{C}(\{\xi\}, \Lambda(\xi)) \rightarrow \mathcal{C}(\Xi, \Lambda)$  is contained in  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$ . Moreover, the induced map  $p_\xi^*: \mathcal{C}(\{\xi\}, \Lambda(\xi)) \rightarrow \mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  is an equivalence of  $\infty$ -categories. In particular,  $\mathcal{C}(\Xi, \Lambda)_{\text{adic}}$  is pre-sentable.*

*Proof.* By the definition of adic objects, to show the first assertion, it suffices to show that the unit transformation  $\text{id}_{\mathcal{C}(\{\xi\}, \Lambda(\xi))} \rightarrow p_{\xi*} p_\xi^*$  is an equivalence. This follows from condition (2b) of Definition 1.1.2 applied to  $p_\xi$ , which is a special case of Lemma 1.3.1. For the second assertion, we only need to show that for every adic complex  $\mathcal{K} \in \mathcal{D}(\Xi, \Lambda)_{\text{adic}}$ , the adjunction map  $p_\xi^* p_{\xi*} \mathcal{K} \rightarrow \mathcal{K}$  is an equivalence. Since  $\prod_{\xi' \in \Xi} e_{\xi'}^*$  is conservative, this is equivalent to showing that  $b: e_{\xi'}^* p_\xi^* p_{\xi*} \mathcal{K} \rightarrow e_{\xi'}^* \mathcal{K}$  is an equivalence for every object  $\xi' \in \Xi$ . Let  $\phi$  be the map  $\xi' \rightarrow \xi$ . Since  $\mathcal{K}$  is adic, the composite

$$\tilde{\phi}^* p_{\xi*} \mathcal{K} \xrightarrow{a} p_{\xi'*} p_{\xi'}^* \tilde{\phi}^* p_{\xi*} \mathcal{K} \simeq p_{\xi'*} i_\phi^* p_\xi^* p_{\xi*} \mathcal{K} \xrightarrow{b} p_{\xi'*} i_\phi^* \mathcal{K}$$

is an equivalence. Moreover, we have shown that  $a$  is an equivalence. Therefore,  $b$  is an equivalence.  $\square$

**Proposition 1.3.4.** *We have*

- (1) *The map  ${}_{\text{chp}^{\text{Ar}}} \text{EO}$  (resp.  ${}_{\text{chp}^{\text{DM}}} \text{EO}$ ) defined above factorizes through the subcategory  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\text{L-tor}})^{op}, \text{Mon}_{\text{pf}}^{\text{Pr}_{\text{st}}^{\text{L}}}(\text{Cat}_\infty))$  (resp.  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\text{tor}})^{op}, \text{Mon}_{\text{pf}}^{\text{Pr}_{\text{st}}^{\text{L}}}(\text{Cat}_\infty))$ ).*
- (2) *The map  ${}_{\text{chp}^{\text{Ar}}} \text{EO}_\otimes^*$  factorizes through the subcategory  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Pr}_{\text{st,cl}}^{\text{L}\otimes})$ .*

*Proof.* By the construction of the above maps by descent and Lemma 1.2.4, we only need to show for (1) that  ${}_{\text{sch}^{\text{qc.sep}}} \text{EO}$  factorizes through  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\text{tor}})^{op}, \text{Mon}_{\text{pf}}^{\text{Pr}_{\text{st}}^{\text{L}}}(\text{Cat}_\infty))$ , and for (2) that  ${}_{\text{sch}^{\text{qc.sep}}} \text{EO}_\otimes^*$  factorizes through  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Pr}_{\text{st,cl}}^{\text{L}\otimes})$ . We check the conditions in Definition 1.1.2: (1a) is automatic; (1b) follows from Lemma 1.3.3; (1c) and (2b) follow from Lemma 1.3.1 and (2a) is then automatic.  $\square$

*Remark 1.3.5.* The arguments of this section show that the functor [18, (2.3)]

$${}_{\mathcal{PT}\text{opos}} \text{EO}_\otimes^*: \mathcal{N}(\mathcal{PT}\text{opos})^{op} \rightarrow \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Pr}_{\text{st,cl}}^{\text{L}\otimes}).$$

${}_{\mathcal{PT}\text{opos}} \text{EO}_\otimes^*$  factorizes through  $\text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Pr}_{\text{st,cl}}^{\text{L}\otimes})$ .

**1.4. Adic complexes as limits.** We put  $\mathcal{A}\text{dic}^\otimes = \mathcal{A}\text{dic}_{\mathcal{R}\text{ing}}^\otimes$ ,  $\underline{\mathcal{A}\text{dic}}^\otimes = \underline{\mathcal{A}\text{dic}}_{\mathcal{R}\text{ing}}^\otimes$ , and  $\alpha^\otimes = \alpha_{\mathcal{R}\text{ing}}^\otimes$ .

*Definition 1.4.1* (Enhanced adic operation map). Define the following four groups of data. Each contains two functors and a natural transformation between them.

$$\begin{aligned}
 \text{Chp}_{\mathbb{L}}^{\text{Ar}} \text{EO} &= \mathcal{A}\text{dic}_{\mathcal{R}\text{ing}_{\mathbb{L}\text{-tor}}} \circ \text{Chp}_{\mathbb{L}}^{\text{Ar}} \text{EO} : \delta_{2,\{2\}}^* \text{Fun}(\Delta^1, \text{Chp}_{\mathbb{L}}^{\text{Ar}})_{F^0, A}^{\text{cart}} \\
 &\rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathbb{L}\text{-tor}})^{\text{op}}, \text{Mon}_{\mathcal{P}\text{f}}^{\mathcal{P}\text{r}_{\text{st}}^{\mathbb{L}}}(\text{Cat}_{\infty})), \\
 \text{Chp}_{\mathbb{L}}^{\text{Ar}} \underline{\text{EO}} &= \underline{\mathcal{A}\text{dic}}_{\mathcal{R}\text{ing}_{\mathbb{L}\text{-tor}}} \circ \text{Chp}_{\mathbb{L}}^{\text{Ar}} \text{EO} : \delta_{2,\{2\}}^* \text{Fun}(\Delta^1, \text{Chp}_{\mathbb{L}}^{\text{Ar}})_{F^0, A}^{\text{cart}} \\
 &\rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\mathbb{L}\text{-tor}})^{\text{op}}, \text{Mon}_{\mathcal{P}\text{f}}^{\mathcal{P}\text{r}_{\text{st}}^{\mathbb{L}}}(\text{Cat}_{\infty})), \\
 \text{Chp}_{\mathbb{L}}^{\text{Ar}} \alpha \text{EO} &= \alpha_{\mathcal{R}\text{ing}_{\mathbb{L}\text{-tor}}} \circ (\text{Chp}_{\mathbb{L}}^{\text{Ar}} \text{EO} \times \text{id}_{\Delta^1}); \\
 \text{Chp}^{\text{DM}} \text{EO} &= \mathcal{A}\text{dic}_{\mathcal{R}\text{ing}_{\text{tor}}} \circ \text{Chp}^{\text{DM}} \text{EO} : \delta_{2,\{2\}}^* \text{Fun}(\Delta^1, \text{Chp}^{\text{DM}})_{F^0, A}^{\text{cart}} \\
 &\rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}}, \text{Mon}_{\mathcal{P}\text{f}}^{\mathcal{P}\text{r}_{\text{st}}^{\mathbb{L}}}(\text{Cat}_{\infty})), \\
 \text{Chp}^{\text{DM}} \underline{\text{EO}} &= \underline{\mathcal{A}\text{dic}}_{\mathcal{R}\text{ing}_{\text{tor}}} \circ \text{Chp}^{\text{DM}} \text{EO} : \delta_{2,\{2\}}^* \text{Fun}(\Delta^1, \text{Chp}^{\text{DM}})_{F^0, A}^{\text{cart}} \\
 &\rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}}, \text{Mon}_{\mathcal{P}\text{f}}^{\mathcal{P}\text{r}_{\text{st}}^{\mathbb{L}}}(\text{Cat}_{\infty})), \\
 \text{Chp}^{\text{DM}} \alpha \text{EO} &= \alpha_{\mathcal{R}\text{ing}_{\text{tor}}} \circ (\text{Chp}^{\text{DM}} \text{EO} \times \text{id}_{\Delta^1}); \\
 \text{Chp}_{\text{Ar}}^{\text{adic}} \text{EO}_{\otimes}^* &= \mathcal{A}\text{dic}_{\otimes} \circ \text{Chp}_{\text{Ar}} \text{EO}_{\otimes}^* : (\text{Chp}_{\text{Ar}})^{\text{op}} \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{\text{op}}, \mathcal{P}\text{r}_{\text{st,cl}}^{\mathbb{L}_{\otimes}}), \\
 \text{Chp}_{\text{Ar}} \underline{\text{EO}}_{\otimes}^* &= \underline{\mathcal{A}\text{dic}}_{\otimes} \circ \text{Chp}_{\text{Ar}} \text{EO}_{\otimes}^* : (\text{Chp}_{\text{Ar}})^{\text{op}} \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{\text{op}}, \mathcal{P}\text{r}_{\text{st,cl}}^{\mathbb{L}_{\otimes}}), \\
 \text{Chp}_{\text{Ar}} \alpha \text{EO}_{\otimes}^* &= \alpha^{\otimes} \circ (\text{Chp}_{\text{Ar}} \text{EO}_{\otimes}^* \times \text{id}_{\Delta^1}); \\
 \mathcal{P}\mathcal{T}\text{opos}^{\text{adic}} \text{EO}_{\otimes}^* &= \mathcal{A}\text{dic}_{\otimes} \circ \mathcal{P}\mathcal{T}\text{opos} \text{EO}_{\otimes}^* : (\mathcal{P}\mathcal{T}\text{opos})^{\text{op}} \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{\text{op}}, \mathcal{P}\text{r}_{\text{st,cl}}^{\mathbb{L}_{\otimes}}), \\
 \mathcal{P}\mathcal{T}\text{opos} \underline{\text{EO}}_{\otimes}^* &= \underline{\mathcal{A}\text{dic}}_{\otimes} \circ \mathcal{P}\mathcal{T}\text{opos} \text{EO}_{\otimes}^* : (\mathcal{P}\mathcal{T}\text{opos})^{\text{op}} \rightarrow \text{Fun}^{\text{top}}(\mathcal{N}(\mathcal{R}\text{ind})^{\text{op}}, \mathcal{P}\text{r}_{\text{st,cl}}^{\mathbb{L}_{\otimes}}), \\
 \mathcal{P}\mathcal{T}\text{opos} \alpha \text{EO}_{\otimes}^* &= \alpha^{\otimes} \circ (\mathcal{P}\mathcal{T}\text{opos} \text{EO}_{\otimes}^* \times \text{id}_{\Delta^1}).
 \end{aligned}$$

By construction and Remark 1.2.3, the six functors in first three groups satisfy (P4).

For objects  $X$  of  $\mathcal{P}\mathcal{T}\text{opos}$  (resp.  $\text{Chp}_{\text{Ar}}^{\text{Ar}}$ ) and  $\lambda = (\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}$ , we write  $\mathcal{D}(X, \lambda)_{\text{adic}}^{\otimes}$  for  $\mathcal{P}\mathcal{T}\text{opos}^{\text{adic}} \text{EO}_{\otimes}^*(X, \lambda)$  (resp.  $\text{Chp}_{\text{Ar}}^{\text{adic}} \text{EO}_{\otimes}^*(X, \lambda)$ ), which is a full (symmetric monoidal) subcategory of  $\mathcal{D}(X, \lambda)^{\otimes}$ . Similarly, we write  $\underline{\mathcal{D}}(X, \lambda)^{\otimes}$  for  $\mathcal{P}\mathcal{T}\text{opos} \underline{\text{EO}}_{\otimes}^*(X, \lambda)$  (resp.  $\text{Chp}_{\text{Ar}} \underline{\text{EO}}_{\otimes}^*(X, \lambda)$ ). Recall that  $\Xi$  is by definition a partially ordered set.

**Proposition 1.4.2.** *The natural transformation  $\mathcal{P}\mathcal{T}\text{opos}^{\alpha} \text{EO}_{\otimes}^*$  is a natural equivalence. In other words, for every object  $T$  of  $\mathcal{P}\mathcal{T}\text{opos}$  and every object  $\lambda = (\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}$ , the symmetric monoidal functor*

$$\mathcal{P}\mathcal{T}\text{opos}^{\alpha} \text{EO}_{\otimes}^*(T, \lambda) : \mathcal{D}(T, \lambda)_{\text{adic}}^{\otimes} \rightarrow \underline{\mathcal{D}}(T, \lambda)^{\otimes} \simeq \varprojlim_{\Xi^{\text{op}}} \mathcal{D}(T, \Lambda(\xi))^{\otimes}$$

*is an equivalence.*

*Proof.* By definition  $\mathcal{D}(T, \lambda)_{\text{adic}}$  is a full subcategory of  $\mathcal{D}(T, \lambda) = \mathcal{D}(\text{Mod}(T_{\text{et}}^{\Xi}, \Lambda))$ . We analyze the construction of the functor  $\alpha : \mathcal{D}(T^{\Xi}, \Lambda)_{\text{adic}} \rightarrow \varprojlim_{\Xi^{\text{op}}} \mathcal{D}(T, \Lambda(\xi))$ . First, we have a functor  $\Delta^1 \times \mathcal{N}(\Xi) \rightarrow \mathcal{P}\text{r}_{\text{st}}^{\mathbb{L}}$  sending  $\Delta^1 \times (\varphi : \xi \rightarrow \xi')$  to the square

$$\begin{array}{ccc}
 \mathcal{D}(T^{\Xi/\xi}, \Lambda_{/\xi}) & \xrightarrow{p_{\xi*}} & \mathcal{D}(T, \Lambda(\xi)) \\
 i_{\varphi*} \downarrow & & \downarrow \tilde{\varphi}_* \\
 \mathcal{D}(T^{\Xi/\xi'}, \Lambda_{/\xi'}) & \xrightarrow{p_{\xi'*}} & \mathcal{D}(T, \Lambda(\xi')).
 \end{array}$$

This corresponds to a projectively fibrant simplicial functor  $\mathcal{F}: \mathfrak{C}[N(D)] \rightarrow \text{Set}_\Delta^+$ , where  $D = [1] \times \Xi$ . Let  $\phi_D: \mathfrak{C}[N(D)] \rightarrow D$  be the canonical equivalence of simplicial categories and put  $\mathcal{F}' = (\text{Fibr}^D \circ St_{\phi_D^{op}}^+ \circ \text{Un}_{N(D)^{op}}^+) \mathcal{F}: D \rightarrow \text{Set}_\Delta^+$ . We write  $\mathcal{F}'$  in the form  $\mathcal{F}': [1] \rightarrow (\text{Set}_\Delta^+)^{\Xi}$ . Applying the marked unstraightening functor  $\text{Un}_\phi^+$  for the weak equivalence of simplicial categories  $\phi: \mathfrak{C}[N(\Xi)^{op}] \rightarrow \Xi^{op}$ , we obtain a morphism  $\tilde{\alpha}: F_1 \rightarrow F_2$  of Cartesian fibrations in the category  $(\text{Set}_\Delta^+)_{/N(\Xi)^{op}}$ . Moreover, by [19, 5.2.2.5], both  $F_1$  and  $F_2$  are *coCartesian* fibrations as well, but  $\tilde{\alpha}$  does *not* send coCartesian edges to coCartesian ones in general. By a similar argument, we have a map

$$\mathcal{D}(T^\Xi, \Lambda) \rightarrow \text{Map}_{N(\Xi)^{op}}^{\text{coCart}}(N(\Xi)^{op}, F_1) := \text{Map}_{N(\Xi)^{op}}^b((N(\Xi)^{op})^\sharp, (F_1, \mathcal{E})),$$

where  $\mathcal{E}$  is the set of coCartesian edges of  $F_1$ . Composing with the obvious inclusion and  $\text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, \tilde{\alpha})$ , we obtain a map  $\alpha': \mathcal{D}(T^\Xi, \Lambda) \rightarrow \text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2)$ . We have the equivalence  $\text{Map}_{N(\Xi)^{op}}^{\text{coCart}}(N(\Xi)^{op}, F_2) \simeq \varprojlim_{\Xi^{op}} \mathcal{D}(T, \Lambda(\xi))$ , and the following pullback diagram

$$\begin{array}{ccc} \mathcal{D}(T^\Xi, \Lambda)_{\text{adic}} & \xrightarrow{\alpha} & \text{Map}_{N(\Xi)^{op}}^{\text{coCart}}(N(\Xi)^{op}, F_2) \\ \downarrow & & \downarrow \\ \mathcal{D}(T^\Xi, \Lambda) & \xrightarrow{\alpha'} & \text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2), \end{array}$$

by the definition of adic objects, where vertical arrows are inclusions. We also note that  $\alpha'$  commutes with small colimits by [19, 5.1.2.2].

Recall that for every object  $\xi$  of  $\Xi$ , we have an exact evaluation functor  $e_\xi^*: \text{Mod}(T^\Xi, \Lambda) \rightarrow \text{Mod}(T, \Lambda(\xi))$  (on the level of Abelian categories) which admits a (right exact) left adjoint  $e_{\xi!}: \text{Mod}(T, \Lambda(\xi)) \rightarrow \text{Mod}(T^\Xi, \Lambda)$ . We define a truncation functor  $t_{\leq \xi}: \text{Mod}(T^\Xi, \Lambda) \rightarrow \text{Mod}(T^\Xi, \Lambda)$  such that for a sheaf  $F_\bullet \in \text{Mod}(T^\Xi, \Lambda)$ ,

$$(t_{\leq \xi} F_\bullet)_{\xi'} = \begin{cases} F_{\xi'} & \text{if } \xi' \leq \xi, \\ 0 & \text{otherwise,} \end{cases}$$

which is exact and admits a right adjoint. Let  $\Delta_{/\Xi}$  be the category of simplices of  $N(\Xi)$  of dimension  $\leq 1$ . Then all  $n$ -cells of  $N(\Delta_{/\Xi})$  are degenerate for  $n \geq 2$ . Define a functor

$$\beta': N(\Delta_{/\Xi}) \rightarrow \text{Fun}(\text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2), \mathcal{D}(T^\Xi, \Lambda))$$

sending a typical subcategory  $\xi \rightarrow (\xi \rightarrow \xi') \leftarrow \xi'$  of  $\Delta_{/\Xi}$  to

$$\mathbf{L}e_{\xi!} \circ \epsilon_\xi \longleftarrow t_{\leq \xi} \circ \mathbf{L}e_{\xi'!} \circ \epsilon_{\xi'} \longrightarrow \mathbf{L}e_{\xi'!} \circ \epsilon_{\xi'},$$

where  $\epsilon_\xi: \text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2) \rightarrow \mathcal{D}(T, \Lambda(\xi))$  is the restriction to the fiber at  $\xi$ . The functor  $\text{Fun}(\alpha', \mathcal{D}(T^\Xi, \Lambda)) \circ \beta'$  extends to a functor  $N(\Delta_{/\Xi})^\triangleright \rightarrow \text{Fun}(\mathcal{D}(T^\Xi, \Lambda), \mathcal{D}(T^\Xi, \Lambda))$  carrying  $(\xi \rightarrow (\xi \rightarrow \xi') \leftarrow \xi')^\triangleleft$  to

$$\begin{array}{ccc} \mathbf{L}e_{\xi!} \circ \epsilon_\xi \circ \alpha' & \xleftarrow{t_{\leq \xi}} \mathbf{L}e_{\xi'!} \circ \epsilon_{\xi'} \circ \alpha' & \xrightarrow{\quad} \mathbf{L}e_{\xi'!} \circ \epsilon_{\xi'} \circ \alpha' \\ & \searrow \quad \downarrow \quad \swarrow & \\ & \text{id,} & \end{array}$$

which induces a natural transformation  $\varinjlim (\beta' \circ \alpha') \simeq (\varinjlim \beta') \circ \alpha' \rightarrow \text{id}$ . One checks that the restriction  $\beta = \varinjlim \beta' | \text{Map}_{N(\Xi)^{op}}^{\text{coCart}}(N(\Xi)^{op}, F_2)$  takes values in  $\mathcal{D}(T^\Xi, \Lambda)_{\text{adic}}$ . We prove that  $\beta \circ \alpha \rightarrow \text{id}$  is an equivalence. Pick an object  $\mathcal{K}$  of  $\mathcal{D}(T^\Xi, \Lambda)_{\text{adic}}$ . We need to show that the diagram



$\beta_{\mathcal{K}}^{\triangleright}: N(\Delta_{/\Xi}^{op})^{\triangleright} \rightarrow \mathcal{D}(T^{\Xi}, \Lambda)$ , depicted as

$$\begin{array}{ccc} \mathbf{L}e_{\xi!} \mathcal{K}_{\xi} & \xleftarrow{t_{\leq \xi}} \mathbf{L}e_{\xi!} \mathcal{K}_{\xi'} & \xrightarrow{\quad} \mathbf{L}e_{\xi!} \mathcal{K}_{\xi'} \\ & \searrow \downarrow \swarrow & \\ & \mathcal{K} & \end{array}$$

is a colimit diagram. We only need to check this after applying  $e_{\xi_0}^*$  for every  $\xi_0 \in \Xi$ , since  $e_{\xi_0}^*$  commutes with colimits. The composite functor  $e_{\xi_0}^* \circ \beta_{\mathcal{K}}^{\triangleright}$  has value (equivalent to)  $\mathcal{K}_{\xi_0}$  on the cone point, vertices  $\{\xi\}$ ,  $(\xi \rightarrow \xi')$  of  $\Delta_{/\Xi}$  for  $\xi \geq \xi_0$  and 0 otherwise, with all morphisms being either identities on  $\mathcal{K}_{\xi_0}$  or 0, or the zero morphism  $0 \rightarrow \mathcal{K}_{\xi_0}$ . It is clear that  $e_{\xi_0}^* \circ \beta_{\mathcal{K}}^{\triangleright}$  induces an equivalence  $\varinjlim (e_{\xi_0}^* \circ \beta_{\mathcal{K}}^{\triangleright} | N(\Delta_{/\Xi}^{op})) \simeq \mathcal{K}_{\xi_0}$  in  $\mathcal{D}(T, \Lambda(\xi_0))$ .

For the other direction, the functor  $\text{Fun}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2), \alpha') \circ \beta'$  also extends to a functor

$$N(\Delta_{/\Xi})^{\triangleright} \rightarrow \text{Fun}(\text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2), \text{Map}_{N(\Xi)^{op}}(N(\Xi)^{op}, F_2))$$

carrying  $(\xi \rightarrow (\xi \rightarrow \xi') \leftarrow \xi')^{\triangleleft}$  to

$$\begin{array}{ccc} \alpha' \circ \mathbf{L}e_{\xi!} \circ \epsilon_{\xi} & \xleftarrow{\alpha' \circ t_{\leq \xi}} \alpha' \circ \mathbf{L}e_{\xi!} \circ \epsilon_{\xi'} & \xrightarrow{\quad} \alpha' \circ \mathbf{L}e_{\xi!} \circ \epsilon_{\xi'} \\ & \searrow \downarrow \swarrow & \\ & \text{id} & \end{array}$$

which induces a natural transformation  $\varinjlim (\alpha' \circ \beta') \simeq \alpha' \circ (\varinjlim \beta') \rightarrow \text{id}$ , where the equivalence of two functors is due to the fact that  $\alpha'$  commutes with colimits. Restricting to  $\text{Map}_{N(\Xi)^{op}}^{\text{coCart}}(N(\Xi)^{op}, F_2)$ , one obtains a map  $\alpha \circ \beta \rightarrow \text{id}$  which is an equivalence by an argument similar to the above. Therefore,  $\alpha$  is an equivalence and the proposition follows.  $\square$

**Corollary 1.4.3.** *The three natural transformations  ${}_{\text{Chp}^{\text{Ar}}} \alpha \text{EO}$ ,  ${}_{\text{Chp}^{\text{DM}}} \alpha \text{EO}$ , and  ${}_{\text{Chp}^{\text{Ar}}} \alpha \text{EO}^*$  are all natural equivalences. In particular, for every object  $X$  of  $\text{Chp}^{\text{Ar}}$  and every object  $\lambda = (\Xi, \Lambda)$  of  $\text{Rind}$ , the symmetric monoidal functor*

$${}_{\text{Chp}^{\text{Ar}}} \alpha \text{EO}^*_{\otimes}(X, \lambda): \mathcal{D}(X, \lambda)_{\text{adic}}^{\otimes} \rightarrow \underline{\mathcal{D}}(X, \lambda)^{\otimes} \simeq \varprojlim_{\Xi^{op}} \mathcal{D}(X, \Lambda(\xi))^{\otimes}$$

*is an equivalence.*

*Proof.* It suffices to show the second assertion. Since smooth surjective morphisms are of both  ${}_{\text{Chp}^{\text{Ar}}} \text{adicEO}^*_{\otimes}$ - and  ${}_{\text{Chp}^{\text{Ar}}} \text{EO}^*_{\otimes}$ -descent, we may assume that  $X$  is an object of  $\text{Sch}^{\text{qc.sep}}$ , that is, a disjoint union of quasi-compact and separated schemes. In this case, it suffices to apply Proposition 1.4.2 to  $T = X_{\text{ét}}$ .  $\square$

**Remark 1.4.4.** In the special case where  $\Lambda: \Xi^{op} \rightarrow \text{Ring}$  is a constant functor with value  $\Lambda$  (by abuse of notation), we have an equivalence

$$(1.3) \quad \mathcal{D}(X, \lambda)_{\text{adic}}^{\otimes} \xrightarrow{\sim} \underline{\mathcal{D}}(X, \lambda)^{\otimes} \simeq \prod_{\pi_0(\Xi)} \mathcal{D}(X, \Lambda)^{\otimes}$$

given by the product functor  $\prod_{\pi_0(\Xi)} e_{\xi_i}^*$ , where we have arbitrarily chosen an object  $\xi_i$  in each connected component  $\Xi_i$  of  $\Xi$ . The resulting functor is independent of such choices up to equivalence.

Assume  $\Xi$  is connected for simplicity. Let  $\pi: (\Xi, \Lambda) \rightarrow (*, \Lambda)$  be the projection. Then  $\pi^*: \mathcal{D}(X, \Lambda)^{\otimes} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}}^{\otimes}$  is an equivalence since its composition with (1.3) is the identity. In particular, the right adjoint functor (between the underlying  $\infty$ -categories)  $\mathbf{R}\pi_* | \mathcal{D}(X, \lambda)_{\text{adic}}: \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \Lambda)$  is an equivalence as well. The special case  $\Xi = \mathbb{N}$  was proved in [14, 2.2.5] assuming the finiteness condition on cohomological dimension.



**1.5. Enhanced six operations and the usual  $t$ -structure.** Recall that [20, 1.4.4.12], for a presentable stable  $\infty$ -category  $\mathcal{D}$ , a  $t$ -structure is *accessible* if the full subcategory  $\mathcal{D}^{\leq 0}$  is presentable. For a topos  $X \in \mathcal{P}\text{Topos}$  or a scheme  $X \in \text{Sch}^{\text{qc.sep}}$ , the usual  $t$ -structure on  $\mathcal{D}(X, \lambda)$  is accessible by [20, 1.3.5.21]. For a higher Artin stack  $X$ , the usual  $t$ -structure on  $\mathcal{D}(X, \lambda)$  is accessible by construction [18, 4.3.7] (Part (2) of (P6)).

Let  $\mathcal{D}^{\leq n}(X, \lambda)_{\text{adic}} = \mathcal{D}^{\leq n}(X, \lambda) \cap \mathcal{D}(X, \lambda)_{\text{adic}}$  and  $\mathcal{D}^{\geq n}(X, \lambda)_{\text{adic}} = \mathcal{D}^{\leq n-1}(X, \lambda)_{\text{adic}}^\perp$ . Recall by Proposition 1.1.6 that the inclusion  $\mathcal{D}(X, \lambda)_{\text{adic}} \subseteq \mathcal{D}(X, \lambda)$  admits a right adjoint, that is, the colocalization functor  $R_X: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}}$ . By Lemma 1.1.5 and [19, 5.5.3.12],  $\mathcal{D}^{\leq n}(X, \lambda)_{\text{adic}}$  is presentable. The inclusion  $\mathcal{D}^{\leq n}(X, \lambda)_{\text{adic}} \subseteq \mathcal{D}(X, \lambda)$  preserves all small colimits and  $\mathcal{D}^{\leq n}(X, \lambda)_{\text{adic}}$  is closed under extension. By [20, 1.4.4.11 (1)], the pair  $(\mathcal{D}^{\leq n}(X, \lambda)_{\text{adic}}, \mathcal{D}^{\geq n}(X, \lambda)_{\text{adic}})$  define an accessible  $t$ -structure, called the usual  $t$ -structure, on  $\mathcal{D}(X, \lambda)_{\text{adic}}$ .

Now we define six operations for adic complexes and study their behavior under the above  $t$ -structure. It is clear that  $- \otimes -$  preserves the subcategory  $\mathcal{D}(X, \lambda)_{\text{adic}}$ . Therefore, we have the induced (derived) tensor product

$$\mathbf{3L}: - \otimes - = - \otimes_X -: \mathcal{D}(X, \lambda)_{\text{adic}} \times \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}},$$

that is left  $t$ -exact with respect to the above  $t$ -structure. Moving the first factor of the source  $\mathcal{D}(X, \lambda)_{\text{adic}} \times \mathcal{D}(X, \lambda)_{\text{adic}}$  to the target side, we can write the functor  $- \otimes -$  in the form  $\mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \text{Fun}^L(\mathcal{D}(X, \lambda)_{\text{adic}}, \mathcal{D}(X, \lambda)_{\text{adic}})$ , because the tensor product on  $\mathcal{D}(X, \lambda)_{\text{adic}}$  is closed. Taking opposites and applying [19, 5.2.6.2], we obtain a functor  $\mathcal{D}(X, \lambda)_{\text{adic}}^{\text{op}} \rightarrow \text{Fun}^R(\mathcal{D}(X, \lambda)_{\text{adic}}, \mathcal{D}(X, \lambda)_{\text{adic}})$ , which can be written as

$$\mathbf{3R}: \underline{\mathbf{Hom}}(-, -) = \underline{\mathbf{Hom}}_X(-, -): \mathcal{D}(X, \lambda)_{\text{adic}}^{\text{op}} \times \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}}.$$

Moreover, we have

$$\underline{\mathbf{Hom}}_X(-, -) \simeq R_X \circ \mathbf{Hom}_X(-, -) | \mathcal{D}(X, \lambda)_{\text{adic}}^{\text{op}} \times \mathcal{D}(X, \lambda)_{\text{adic}}.$$

For every morphism  $\pi: \lambda' \rightarrow \lambda$  of  $\text{Rind}$ ,  $\pi^*$  preserves adic objects by Lemma 1.1.3 (1), so that we have

$$\underline{\pi}^* = \pi^* | \mathcal{D}(X, \lambda): \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda')_{\text{adic}},$$

which admits a right adjoint

$$\underline{\pi}_* \simeq R_X \circ \pi_* | \mathcal{D}(X, \lambda'): \mathcal{D}(X, \lambda')_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}}.$$

Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks. Then  $f^*$  preserves adic objects by Lemma 1.1.3 (2). Therefore, we have the induced functor

$$\mathbf{1L}: \underline{f}_* = f^* | \mathcal{D}(X, \lambda)_{\text{adic}}: \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(Y, \lambda)_{\text{adic}},$$

which admits a right adjoint, denoted by

$$\mathbf{1R}: \underline{f}_* \simeq R_X \circ f_* | \mathcal{D}(Y, \lambda)_{\text{adic}}: \mathcal{D}(Y, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}}.$$

It follows from the definition and the corresponding properties of  $f^*$  and  $f_*$ , that  $\underline{f}_*$  is right  $t$ -exact (i.e. preserves  $\mathcal{D}^{\leq 0}(-)_{\text{adic}}$ ), and  $\underline{f}_*$  is left  $t$ -exact (i.e. preserves  $\mathcal{D}^{\geq 0}(-)_{\text{adic}}$ ). For  $f$  surjective and  $\mathcal{K} \in \mathcal{D}(X, \lambda)$ ,  $\mathcal{K}$  is adic if and only if  $f^*\mathcal{K}$  is adic.

If  $f$  is a morphism, locally of finite type, of higher Artin stacks in  $\text{Chp}_L^{\text{Ar}}$  (resp. of higher Deligne–Mumford stacks, resp. locally quasi-finite of higher Deligne–Mumford stacks) and  $\lambda$  is an object of  $\text{Rind}_L\text{-tor}$  (resp.  $\text{Rind}_{\text{tor}}$ , resp.  $\text{Rind}$ ), then  $f_!$  preserves adic objects. Therefore, we have the induced functor

$$\mathbf{2L}: \underline{f}_! = f_! | \mathcal{D}(Y, \lambda)_{\text{adic}}: \mathcal{D}(Y, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(X, \lambda)_{\text{adic}},$$

which admits a right adjoint, denoted by

$$\mathbf{2R}: \underline{f}^! \simeq R_Y \circ f^! | \mathcal{D}(X, \lambda)_{\text{adic}}: \mathcal{D}(X, \lambda)_{\text{adic}} \rightarrow \mathcal{D}(Y, \lambda)_{\text{adic}}.$$

It follows from the definition and the corresponding properties of  $f_!$  and  $f^!$  [18, Lemma 6.1.13], that  $\underline{f}_![2d]$  (resp.  $\underline{f}^![-2d]$ ) is right (resp. left)  $t$ -exact if  $d = \dim^+(f) < \infty$ .

If  $f: Y \rightarrow X$  is a morphism, locally of finite type, of locally Noetherian and locally finite-dimensional higher Artin stacks in  $\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}$ , and if  $\lambda$  is an object of  $\mathrm{Rind}_{\mathbb{L}\text{-tor}}$ , then  $f_*$  and  $f^!$  preserve adic objects by [18, Proposition 6.2.1], so that  $\underline{f}_*$  and  $\underline{f}^!$  are restrictions of  $f_*$  and  $f^!$ .

The  $t$ -structure on  $\mathcal{D}(X, \lambda)$  naturally induces a  $t$ -structure on  $\underline{\mathcal{D}}(X, \lambda)$ . More precisely, for every integer  $n$ , an object  $\mathcal{K}$  of  $\mathcal{D}(X, \lambda) = \varprojlim_{\Xi \text{ op}} \mathcal{D}(X, \Lambda(\xi))$  is in  $\underline{\mathcal{D}}^{\leq n}(X, \lambda)$  if and only if  $\mathcal{K}_\xi$  is in  $\mathcal{D}^{\leq n}(X, \Lambda(\xi))$  for all objects  $\xi$  of  $\Xi$ . Here,  $\mathcal{K}_\xi$  is the image of  $\mathcal{K}$  under the natural functor  $\varprojlim_{\Xi \text{ op}} \mathcal{D}(X, \Lambda(\xi)) \rightarrow \mathcal{D}(X, \Lambda(\xi))$ . Let  $\underline{\mathcal{D}}^{\geq n}(X, \lambda) = \underline{\mathcal{D}}^{\leq n-1}(X, \lambda)^\perp$ . We have induced functors and properties similar to those for  $\mathcal{D}(X, \lambda)_{\mathrm{adic}}$ . The equivalence  ${}_{\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}} \mathrm{EO}^*(X, \lambda)$  is compatible with the two  $t$ -structures on  $\mathcal{D}(X, \lambda)_{\mathrm{adic}}$  and  $\underline{\mathcal{D}}(X, \lambda)$ , as well as the various induced functors.

In what follows, we will identify  $\mathcal{D}(X, \lambda)_{\mathrm{adic}}$  and  $\underline{\mathcal{D}}(X, \lambda)$  (as stable  $\infty$ -categories with  $t$ -structures) and the corresponding functors. We will use the *underlined* version in all the notation. In particular, we will view  $\underline{\mathcal{D}}(X, \lambda)$  as a (strictly) full subcategory of  $\mathcal{D}(X, \lambda)$ .

**Proposition 1.5.1.** *The six operations and the usual  $t$ -structure in the above adic setting have all the properties stated in [18, Propositions 6.1.1, 6.1.2, 6.1.3, 6.1.4, 6.1.11, 6.1.12, Corollary 6.1.10].*

*Proof.* (The analogues of) Propositions [18, 6.1.1, 6.1.2] are encoded in  ${}_{\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}} \underline{\mathrm{EO}} \simeq {}_{\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}} \mathrm{EO}^{\mathrm{adic}}$  and  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \underline{\mathrm{EO}} \simeq {}_{\mathrm{Chp}^{\mathrm{DM}}} \mathrm{EO}^{\mathrm{adic}}$ . Propositions [18, 6.1.3, 6.1.4] follow from the previous two propositions. Proposition [18, 6.1.11] follows by restricting to the full subcategory spanned by adic objects, since  $f_*$  preserves adic objects in this case. Proposition [18, 6.1.12] follows from property (P4) of  ${}_{\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}} \underline{\mathrm{EO}}$  and  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \underline{\mathrm{EO}}$ . Corollary [18, 6.1.10] follows from Proposition [18, 6.1.1] and Proposition 1.5.2 (2).  $\square$

With slight modification as follows, Proposition [18, 6.1.9] also holds in the adic setting by restriction to the full subcategory of adic objects.

**Proposition 1.5.2** (Poincaré duality). *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a flat (resp. flat and locally quasi-finite) morphism of  $\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}^{\mathrm{DM}}$ ), locally of finite presentation. Let  $\lambda$  be an object of  $\mathrm{Rind}_{\mathbb{L}\text{-tor}}$  (resp.  $\mathrm{Rind}$ ). Then*

- (1) *There is a natural transformation  $\underline{u}_f: \underline{f}_! \circ \underline{f}^* \langle d \rangle \rightarrow \mathrm{id}_{\mathcal{X}}$  for every integer  $d \geq \dim^+(f)$ .*
- (2) *If  $f$  is moreover smooth, then the induced natural transformation  $\underline{u}_f: \underline{f}_! \circ \underline{f}^* \langle \dim f \rangle \rightarrow \mathrm{id}_{\mathcal{X}}$  is a counit transformation, so that the induced map  $\underline{f}^* \langle \dim f \rangle \rightarrow \underline{f}^!$  is a natural equivalence of functors  $\underline{\mathcal{D}}(X, \lambda) \rightarrow \underline{\mathcal{D}}(Y, \lambda)$ .*

*Remark 1.5.3.* By Poincaré duality,  $\underline{f}^*$  is  $t$ -exact if  $f$  is a smooth morphism of higher Artin stacks in  $\mathrm{Chp}_{\mathbb{L}}^{\mathrm{Ar}}$  and  $\lambda$  is an object of  $\mathrm{Rind}_{\mathbb{L}\text{-tor}}$  or if  $f$  is an étale morphism of higher Deligne–Mumford stacks. We will show in §2.3 that this holds for more general morphisms  $f$  under a finiteness assumption.

**Lemma 1.5.4.** *Let  $(\Xi, \Lambda)$  be an object of  $\mathrm{Rind}$  and  $\xi$  be an object of  $\Xi$ ,  $\mathcal{K}$  be an adic object of  $\mathcal{D}(X, (\Xi, \Lambda))$ . The following diagram*

$$\begin{array}{ccc} \mathcal{D}(X, (\Xi, \Lambda)) & \xleftarrow{-\otimes \mathcal{K}} & \mathcal{D}(X, (\Xi, \Lambda)) \\ e_\xi^* \downarrow & & \downarrow e_\xi^* \\ \mathcal{D}(X, \Lambda(\xi)) & \xleftarrow{-\otimes e_\xi^* \mathcal{K}} & \mathcal{D}(X, \Lambda(\xi)) \end{array}$$

*is right adjointable and its transpose is left adjointable. In other words, the natural morphisms  $e_{\xi!}(\mathcal{L} \otimes e_\xi^* \mathcal{K}) \rightarrow (e_{\xi!} \mathcal{L}) \otimes \mathcal{K}$  and  $e_\xi^* \mathbf{Hom}(\mathcal{K}, \mathcal{L}') \rightarrow \mathbf{Hom}(e_\xi^* \mathcal{K}, e_\xi^* \mathcal{L}')$  are equivalences for objects  $\mathcal{L}$  of  $\mathcal{D}(X, \Lambda(\xi))$  and  $\mathcal{L}'$  of  $\mathcal{D}(X, (\Xi, \Lambda))$ .*

*Proof.* By [18, 6.1.7], we may assume that  $\xi$  is the final object of  $\Xi$ . In this case,  $e_\xi^*$  can be identified with  $\pi_*$ , where  $\pi: (\Xi, \Lambda) \rightarrow (\{\xi\}, \Lambda(\xi))$ . Since  $\mathcal{K}$  is adic, the morphism  $\pi^* e_\xi^* \mathcal{K} \rightarrow \mathcal{K}$  is an equivalence. A left adjoint of the transpose of the above diagram is then given by the diagram

$$\begin{array}{ccc} \mathcal{D}(X, (\Xi, \Lambda)) & \xleftarrow{\pi^*} & \mathcal{D}(X, \Lambda(\xi)) \\ -\otimes \mathcal{K} \downarrow & & \downarrow -\otimes e_\xi^* \mathcal{K} \\ \mathcal{D}(X, (\Xi, \Lambda)) & \xleftarrow{\pi^*} & \mathcal{D}(X, \Lambda(\xi)). \end{array}$$

□

**1.6. Adic dualizing complexes.** Let  $X$  be an object of  $\mathbf{Chp}^{\text{Ar}}$  and  $\lambda = (\Xi, \Lambda)$  be an object of  $\mathbf{Rind}$ . Let  $\mathcal{O}$  be an objects of  $\mathcal{D}(X, \lambda)$  (resp.  $\underline{\mathcal{D}}(X, \lambda)$ ). By adjunction of the pair of functors  $-\otimes \mathcal{K}$  and  $\mathbf{Hom}(\mathcal{K}, -)$  (resp.  $-\otimes \mathcal{K}$  and  $\underline{\mathbf{Hom}}(\mathcal{K}, -)$ ), we have a natural transformation  $\delta_{\mathcal{O}}: \text{id} \rightarrow \mathbf{hHom}(\mathbf{hHom}(-, \mathcal{O}), \mathcal{O})$  (resp.  $\underline{\delta}_{\mathcal{O}}: \text{id} \rightarrow \mathbf{hHom}(\mathbf{hHom}(-, \mathcal{O}), \mathcal{O})$ ) between endofunctors of  $\mathbf{hD}(X, \lambda)$  (resp.  $\mathbf{h}\underline{\mathcal{D}}(X, \lambda)$ ), which is called the *biduality transformation*<sup>2</sup>.

In the remainder of this section, we fix an  $\mathbf{L}$ -coprime base scheme  $\mathbf{S}$  that is a disjoint union of *excellent schemes*<sup>3</sup>, endowed with a *global dimension function*. Let  $\mathbf{Rind}_{\mathbf{L}\text{-dual}}$  be the full subcategory of  $\mathbf{Rind}_{\mathbf{L}\text{-tor}}$  spanned by ringed diagrams  $\Lambda: \Xi^{\text{op}} \rightarrow \mathbf{Ring}$  such that  $\Lambda(\xi)$  is an ( $\mathbf{L}$ -torsion) Gorenstein ring of dimension 0 for every object  $\xi$  of  $\Xi$ .

*Definition 1.6.1* (Potential dualizing complex). Let  $\lambda = (\Xi, \Lambda)$  be an object of  $\mathbf{Rind}_{\mathbf{L}\text{-dual}}$ . For an object  $f: X \rightarrow \mathbf{S}$  of  $\mathbf{Chp}_{\text{lift}/\mathbf{S}}^{\text{Ar}}$  with  $X$  in  $\mathbf{Sch}^{\text{qc.sep}}$ , we say a complex  $\mathcal{O}$  in  $\mathcal{D}(X, \lambda)$  is a *pinned/potential dualizing complex* (on  $X$ ) if

- (1)  $\mathcal{O}$  is adic, and
- (2) for every object  $\xi$  of  $\Xi$ ,  $\mathcal{O}_\xi = e_\xi^* \mathcal{O} \in \mathcal{D}(X, \Lambda(\xi))$  is a pinned/potential dualizing complex.

For a general object  $f: X \rightarrow \mathbf{S}$  of  $\mathbf{Chp}_{\text{lift}/\mathbf{S}}^{\text{Ar}}$ , we say a complex  $\mathcal{O}$  in  $\mathcal{D}(X, \lambda)$  is a *pinned/potential dualizing complex* if for every atlas  $u: X_0 \rightarrow X$  with  $X_0$  in  $\mathbf{Sch}^{\text{qc.sep}}$ ,  $u^! \mathcal{O}$  is a *pinned/potential dualizing complex* on  $X_0$ .

**Proposition 1.6.2.** *Let  $f: X \rightarrow \mathbf{S}$  be an object of  $\mathbf{Chp}_{\text{lift}/\mathbf{S}}^{\text{Ar}}$  and  $\lambda$  be as above. The full subcategory of  $\mathcal{D}(X, \lambda)$  spanned by all pinned/potential dualizing complexes is equivalent to the nerve of an ordinary category consisting of only one object  $\mathcal{O}$  with  $\text{Hom}(\mathcal{O}, \mathcal{O}) = \left( \varprojlim_{\xi \in \Xi} \Lambda(\xi) \right)^{\pi_0(X)}$ . Moreover, pinned/potential dualizing complexes are constructible and compatible under extension of scalars.*

In the proof, we will use the following observation which is essentially [19, A.3.2.27]. Let  $\mathcal{C}: K^\triangleleft \rightarrow \mathbf{Cat}_\infty$  be a functor that is a limit diagram. Let  $X, Y$  be two objects in the limit  $\infty$ -category  $\mathcal{C}_\infty$  and write  $X_k, Y_k$  the natural images in  $\mathcal{C}_k$  for every vertex  $k$  of  $K$ . Then  $\text{Map}_{\mathcal{C}_\infty}(X, Y)$  is naturally the homotopy limit (in the  $\infty$ -category  $\mathcal{H}$  of spaces) of a diagram  $K \rightarrow \mathcal{H}$  sending  $k$  to  $\text{Map}_{\mathcal{C}_k}(X_k, Y_k)$ .

*Proof.* When  $\Xi = *$  is a singleton and  $X$  is in  $\mathbf{Sch}^{\text{qc.sep}}$ , the proposition is proved in [10] (see [18, 6.3.3 (1)]). We also note that if  $\mathcal{O}_{\mathbf{S}}$  is a pinned dualizing complex on  $\mathbf{S}$ , then  $f^! \mathcal{O}_{\mathbf{S}}$  is a pinned dualizing complex on  $X$ . First, we prove by induction on  $k$  that for an object  $f: X \rightarrow \mathbf{S}$  of  $\mathbf{Chp}_{\text{lift}/\mathbf{S}}^{\text{Ar}}$  with  $X$  in  $\mathbf{Chp}^{k\text{-Ar}}$ ,

- (1) For any two pinned dualizing complexes  $\mathcal{O}$  and  $\mathcal{O}'$ ,  $\text{Map}_{\mathcal{D}(X, \Lambda)}(\mathcal{O}, \mathcal{O}')$  is discrete;
- (2) There is a unique distinguished equivalence  $o: \mathcal{O} \rightarrow \mathcal{O}'$  such that for every atlas  $u: X_0 \rightarrow X$  with  $X_0$  in  $\mathbf{Sch}^{\text{qc.sep}}$ ,  $u^! o$  is the one preserving pinning.

<sup>2</sup>In fact,  $\delta_{\mathcal{O}}$  can be enhanced to a natural transformation  $\tilde{\delta}_{\mathcal{O}}: \text{id} \rightarrow \mathbf{Hom}(\mathbf{Hom}(-, \mathcal{O}), \mathcal{O})$  (resp.  $\tilde{\underline{\delta}}_{\mathcal{O}}: \text{id} \rightarrow \underline{\mathbf{Hom}}(\underline{\mathbf{Hom}}(-, \mathcal{O}), \mathcal{O})$ ) between endofunctors of  $\mathcal{D}(X, \lambda)$  (resp.  $\underline{\mathcal{D}}(X, \lambda)$ ), that is,  $\text{h}\tilde{\delta}_{\mathcal{O}} = \delta_{\mathcal{O}}$ . We omit the details here since we do not need such enhancement in what follows.

<sup>3</sup>A scheme is *excellent* if it is quasi-compact and admits a Zariski open cover by spectra of excellent rings [1, 7.8.2].

It is clear that once the equivalence  $o$  in (2) exists, it is compatible under  $f^!$  for every smooth morphism  $f$ . Choose an atlas  $u: Y \rightarrow X$  (with  $Y$  in  $\mathbf{Chp}^{(k-1)\text{-Ar}}$ ). Since  $u$  is of universal  $\mathbf{Chp}^{\text{Ar}}\mathbf{EO}^!$ -descent, both (1) and (2) follow from the induction hypothesis, the above observation, and the fact that limit of  $k$ -truncated spaces is  $k$ -truncated (which follows from [19, 5.5.6.5]). Second, we show that  $\mathrm{Map}_{\mathcal{D}(X,\Lambda)}(\mathcal{O}, \mathcal{O}) \simeq \pi_0 \mathrm{Map}_{\mathcal{D}(X,\Lambda)}(\mathcal{O}, \mathcal{O})$  is isomorphic to  $\Lambda^{\pi_0(X)}$ . Without loss of generality, we assume that  $X$  is connected. Choose an atlas  $u = \coprod_I u_i: \coprod_I Y_i \rightarrow X$  with  $Y_i$  in  $\mathbf{Sch}^{\mathrm{qc.sep}}$  and is connected. We have the following commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\alpha} & \pi_0 \mathrm{Map}_{\mathcal{D}(X,\Lambda)}(\mathcal{O}, \mathcal{O}) \\ \parallel & & \downarrow \beta \\ \Lambda & \longrightarrow & \bigoplus_I \pi_0 \mathrm{Map}_{\mathcal{D}(Y_i,\Lambda)}(u_i^! \mathcal{O}, u_i^! \mathcal{O}). \end{array}$$

Since  $u^!$  is conservative,  $\beta$  is injective. Since  $\Lambda \rightarrow \pi_0 \mathrm{Map}_{\mathcal{D}(Y_i,\Lambda)}(u_i^! \mathcal{O}, u_i^! \mathcal{O})$  is an isomorphism for every  $i \in I$ ,  $\alpha$  is injective. If we write elements of  $\bigoplus_I \pi_0 \mathrm{Map}_{\mathcal{D}(Y_i,\Lambda)}(u_i^! \mathcal{O}, u_i^! \mathcal{O})$  in the coordinate form  $(\dots, \lambda_i, \dots)$  with respect to the basis consisting of distinguished equivalences, then the image of  $u^!$  must belong to the diagonal since  $X$  is connected. Therefore,  $\alpha$  is an isomorphism. The fact that pinned dualizing complexes are constructible and compatible under extension of scalars follows from the case of schemes.

Now we consider the general coefficient  $\lambda = (\Xi, \Lambda)$ . First, we construct a pinned dualizing complex  $\mathcal{O}_{S,\lambda}$  on the base scheme  $S$ . Let  $\Delta_{/\Xi}$  be the category of simplices of  $\Xi$ . Then all  $n$ -simplices of  $N(\Delta_{/\Xi})$  are degenerate for  $n \geq 2$ . For every object  $\xi$  of  $\Xi$ , denote by  $\mathcal{O}_{S,\xi}$  the pinned dualizing complex in  $\mathcal{D}(S, \Lambda(\xi))$ . We will use the functors  $e_{\xi}^!$  and  $t_{\leq \xi}$  introduced in the proof of Proposition 1.4.2. Define a functor  $\delta: N(\Delta_{/\Xi}) \rightarrow \mathcal{D}(S, \lambda)$  sending a typical subcategory  $\xi \leftarrow (\xi \leq \xi') \rightarrow \xi'$  of  $\Delta_{/\Xi}$  to

$$\mathbf{L}e_{\xi}^! \mathcal{O}_{S,\xi} \longleftarrow \mathbf{L}e_{\xi}^! E_{\xi \leq \xi'} \mathcal{O}_{S,\xi'} \simeq t_{\leq \xi} \mathbf{L}e_{\xi'}^! \mathcal{O}_{S,\xi'} \longrightarrow \mathbf{L}e_{\xi'}^! \mathcal{O}_{S,\xi'},$$

where the left arrow is given by the distinguished equivalence  $E_{\xi \leq \xi'} \mathcal{O}_{S,\xi'} \xrightarrow{\sim} \mathcal{O}_{S,\xi}$ . It is easy to see that  $\mathcal{O}_{S,\lambda} = \varprojlim \delta$ , viewed as an element in  $\mathcal{D}(S, \lambda)$ , satisfies the two requirements in Definition 1.6.1, hence is a pinned dualizing complex. For an object  $f: X \rightarrow S$  of  $\mathbf{Chp}_{\mathrm{lift}/S}^{\mathrm{Ar}}$ , let  $\mathcal{O}_{f,\lambda} = f^! \mathcal{O}$ . Then it is a pinned dualizing complex on  $X$ . The rest of the proposition follows from the fact that  $\mathcal{O}_{f,\lambda}$  is adic, Proposition 1.4.3, the observation before the proof, and the same assertion when  $\Xi$  is a singleton.  $\square$

In what follows, we write  $\mathcal{D} = \mathcal{D}_X = \mathbf{Hom}(-, \Omega_{X,\lambda})$ ,  $\mathbf{D} = \mathbf{D}_X = \mathbf{hD}_X = \mathbf{hHom}(-, \Omega_{X,\lambda})$ , and similarly for  $\underline{\mathcal{D}}$ ,  $\underline{\mathbf{D}}$ .

**Proposition 1.6.3.** *Let  $\mathcal{K} \in \underline{\mathcal{D}}(X, \lambda)$  such that  $\delta_{\Omega_{X,\Lambda(\xi)}}(e_{\xi}^* \mathcal{K})$  is an equivalence for every object  $\xi$  of  $\Xi$ . Then  $\underline{\delta}_{\Omega_{X,\lambda}}(\mathcal{K})$  is an equivalence as well.*

The assumption of the proposition is satisfied for  $\mathcal{K} \in \underline{\mathcal{D}}(X, \lambda) \cap \mathcal{D}_{\mathrm{cons}}^{(b)}(X, \lambda)$ .

*Proof.* We need to show that the natural morphism  $\mathcal{K} \rightarrow \underline{\mathbf{D}} \underline{\mathbf{D}} \mathcal{K}$  is an isomorphism (in the homotopy category of  $\underline{\mathcal{D}}(X, \lambda)$ ). By definition,

$$\begin{aligned} \underline{\mathbf{D}} \underline{\mathbf{D}} \mathcal{K} &= \mathbf{hHom}(\mathcal{K}, \mathbf{hHom}(\mathcal{K}, \Omega_{X,\lambda})) \\ &\simeq \mathbf{h}R_X \mathbf{hHom}(\mathcal{K}, \mathbf{h}R_X \mathbf{hHom}(\mathcal{K}, \Omega_{X,\lambda})) \\ &\simeq \mathbf{h}R_X \mathbf{hHom}(\mathcal{K}, \mathbf{hHom}(\mathcal{K}, \Omega_{X,\lambda})). \end{aligned}$$

It suffices to show that  $\delta_{\Omega_{X,\lambda}}(\mathcal{K}): \mathcal{K} \rightarrow \mathbf{h}\mathbf{Hom}(\mathcal{K}, \mathbf{h}\mathbf{Hom}(\mathcal{K}, \Omega_{X,\lambda}))$  is an equivalence. In fact, since  $\mathcal{K}$  is adic, we have

$$\begin{aligned} e_\xi^* \mathbf{h}\mathbf{Hom}(\mathcal{K}, \mathbf{h}\mathbf{Hom}(\mathcal{K}, \Omega_{X,\lambda})) &\simeq \mathbf{h}\mathbf{Hom}(e_\xi^* \mathcal{K}, \mathbf{h}\mathbf{Hom}(e_\xi^* \mathcal{K}, e_\xi^* \Omega_{X,\lambda})) \\ &\simeq \mathbf{h}\mathbf{Hom}(e_\xi^* \mathcal{K}, \mathbf{h}\mathbf{Hom}(e_\xi^* \mathcal{K}, \Omega_{X,\Lambda(\xi)})) \end{aligned}$$

for every object  $\xi \in \Xi$  by Lemma 1.5.4, which is equivalent to  $e_\xi^* \mathcal{K}$  by the assumption.  $\square$

## 2. THE $\mathfrak{m}$ -ADIC FORMALISM AND CONSTRUCTIBILITY

In this chapter, we introduce a special case of the adic formalism, namely, the  $\mathfrak{m}$ -adic formalism on which there is a good notion of constructibility. Such formalism is enough for most applications. The basic notion of the  $\mathfrak{m}$ -adic formalism is given in §2.1. In §2.3, we introduce some finiteness conditions under which we may refine the construction of the usual  $t$ -structure. Then we define the category of constructible adic complexes in this setting in §2.2, and on which the constructible adic perverse  $t$ -structure in §3.4. The last section §2.5 is dedicated to proving the compatibility between our theory and Laszlo–Olsson [14, 15] under their restrictions.

### 2.1. The $\mathfrak{m}$ -adic formalism.

*Definition 2.1.1.* Define a category  $\mathcal{P}\mathbf{Ring}$  as follows. The objects are pairs  $(\Lambda, \mathfrak{m})$ , where  $\Lambda$  is a small ring and  $\mathfrak{m} \subseteq \Lambda$  is a principal ideal, such that

- $\mathfrak{m}$  is generated by an element that is not a zero divisor;
- the natural homomorphism  $\Lambda \rightarrow \varprojlim_n \Lambda_n$  is an isomorphism, where  $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$  ( $n \in \mathbb{N}$ ).

A morphism from  $(\Lambda', \mathfrak{m}')$  to  $(\Lambda, \mathfrak{m})$  is a ring homomorphism  $\phi: \Lambda' \rightarrow \Lambda$  satisfying  $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}'$ . We denote by  $\mathcal{P}\mathbf{Ring}_{\mathbf{L}\text{-tor}} \subseteq \mathcal{P}\mathbf{Ring}$  the full subcategory spanned by  $(\Lambda, \mathfrak{m})$  such that  $(\mathbb{N}, \Lambda_\bullet) \in \mathcal{R}\mathbf{ind}_{\mathbf{L}\text{-tor}}$ .

We have a natural functor  $\mathcal{P}\mathbf{Ring} \rightarrow \mathbf{Fun}([1], \mathcal{R}\mathbf{ind})$  sending  $(\Lambda, \mathfrak{m})$  to  $(\mathbb{N}, \Lambda_\bullet) \xrightarrow{\pi} (*, \Lambda)$ . In what follows, we simply write  $\Lambda_\bullet$  for the ringed diagram  $(\mathbb{N}, \Lambda_\bullet)$ .

Let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\mathbf{Ring}$ . Let  $X$  be a higher Artin stack. We have a pair of adjoint functors

$$\mathbf{L}\pi^* = \mathbf{L}\pi_X^*: \mathcal{D}(X, \Lambda) \rightarrow \mathcal{D}(X, \Lambda_\bullet); \quad \mathbf{R}\pi_* = \mathbf{R}\pi_{X*}: \mathcal{D}(X, \Lambda_\bullet) \rightarrow \mathcal{D}(X, \Lambda).$$

As  $\pi$  is perfect in the sense of [18, 2.2.8],  $\mathbf{L}\pi^*$  admits a left adjoint [18, 6.1.6].

*Definition 2.1.2* (Normalized complex). A complex  $\mathcal{K} \in \mathcal{D}(X, \Lambda_\bullet)$  is called *normalized* if the cofiber<sup>4</sup> [20, 1.1.1.6] of the adjunction map  $\mathbf{L}\pi^* \mathbf{R}\pi_* \mathcal{K} \rightarrow \mathcal{K}$  is 0. We denote by  $\mathcal{D}_n(X, \Lambda_\bullet)$  the full subcategory of  $\mathcal{D}(X, \Lambda_\bullet)$  spanned by normalized.

The subcategory  $\mathcal{D}_n(X, \Lambda_\bullet) \subseteq \mathcal{D}(X, \Lambda_\bullet)$  is a stable subcategory stable under small limits.

Note that  $\mathcal{D}(X, \Lambda) = \underline{\mathcal{D}}(X, \Lambda)$ , so that the image of  $\mathbf{L}\pi^* = \underline{\pi}^*$  is contained in  $\underline{\mathcal{D}}(X, \Lambda_\bullet)$ . In particular, we have  $\mathcal{D}_n(X, \Lambda_\bullet) \subseteq \underline{\mathcal{D}}(X, \Lambda_\bullet)$ . For the other direction, we have the following result. We define  $\underline{\mathcal{D}}^{(+)}(X, \Lambda_\bullet) = \underline{\mathcal{D}}(X, \Lambda_\bullet) \cap \mathcal{D}^{(+)}(X, \Lambda_\bullet)$ .

**Lemma 2.1.3.** *We have  $\underline{\mathcal{D}}^{(+)}(X, \Lambda_\bullet) \subseteq \mathcal{D}_n(X, \Lambda_\bullet)$ .*

*Proof.* The proof is similar to [26, 4.13].  $\square$

As  $\otimes, f^*, f_!$  preserve adic complexes, these operations preserve normalized complexes in  $\mathcal{D}^{(+)}$ . Next we examine effects of  $\mathbf{Hom}, f_*, f^!$  on normalized complexes, which imply that the restrictions of  $\mathbf{Hom}, \underline{f}_*, \underline{f}^!$  to  $\underline{\mathcal{D}}^{(+)}$  coincide with  $\mathbf{Hom}, f_*, f^!$ .

<sup>4</sup>The underlying object in the ordinary triangulated category is a cone [20, 1.1.2.10].

**Proposition 2.1.4.** *Let  $X$  be a higher Artin (resp. higher Deligne–Mumford) stack and let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}_{\mathbf{L}\text{-tor}}$  (resp.  $\mathcal{P}\text{Ring}$ ). For  $\mathcal{K}, \mathcal{L} \in \mathcal{D}_{\mathbf{n}}(X, \Lambda_{\bullet})$ , and more generally for  $\mathcal{K}, \mathcal{L}$  in the essential image of  $\mathbf{L}\pi^*$ ,  $\mathbf{Hom}(\mathcal{K}, \mathcal{L})$  is adic. In particular,  $\mathbf{Hom}$  restricts to*

$$\mathbf{Hom}: \underline{\mathcal{D}}^{(+)}(X, \Lambda_{\bullet})^{\text{op}} \times \underline{\mathcal{D}}^{(+)}(X, \Lambda_{\bullet}) \rightarrow \underline{\mathcal{D}}(X, \Lambda_{\bullet}).$$

*Proof.* By Poincaré duality, we reduce to the case of schemes. This case is essentially proved in [26, 4.18].  $\square$

**Proposition 2.1.5.** *Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks and let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}$ . Then  $f_*: \mathcal{D}(Y, \Lambda_{\bullet}) \rightarrow \mathcal{D}(X, \Lambda_{\bullet})$  preserves normalized complexes. In particular,  $f_*$  restricts to*

$$f_*: \underline{\mathcal{D}}^{(+)}(Y, \Lambda_{\bullet}) \rightarrow \underline{\mathcal{D}}^{(+)}(X, \Lambda_{\bullet}).$$

*Proof.* This follows from the fact that  $f_*$  commutes with  $\mathbf{L}\pi^*$  [18, 6.1.6].  $\square$

**Proposition 2.1.6.** *Let  $f: Y \rightarrow X$  be a morphism locally of finite type of  $\mathcal{C}\text{hp}_{\mathbf{L}}^{\text{Ar}}$  and let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}_{\mathbf{L}\text{-tor}}$ . Then  $f^!: \mathcal{D}(X, \Lambda_{\bullet}) \rightarrow \mathcal{D}(Y, \Lambda_{\bullet})$  preserves normalized complexes in  $\mathcal{D}^{(+)}$ . In particular,  $f^!$  restricts to*

$$f^!: \underline{\mathcal{D}}^{(+)}(X, \Lambda_{\bullet}) \rightarrow \underline{\mathcal{D}}^{(+)}(Y, \Lambda_{\bullet}).$$

*Proof.* By Poincaré duality applied to atlases, we are reduced to the case of a closed immersion of schemes, which follows from the fact that  $f^!$  commutes with  $\mathbf{L}\pi^*$  [18, 6.1.7] (with no restriction on  $\Lambda$  in this case).  $\square$

The truncation functors  $\tau^{\leq n}, \tau^{\geq n}$  do not preserve normalized complexes in general. In the rest of §2.1, we study the effects of the truncation functors on normalized complexes.

Let  $\mathcal{A}$  be an Abelian category. An object  $M_{\bullet}$  in  $\text{Fun}(\mathbb{N}^{\text{op}}, \mathcal{A})$  is called *essentially null* if for each  $n \in \mathbb{N}$ , there is an  $r \in \mathbb{N}$  such that  $M_{r+n} \rightarrow M_n$  is the zero morphism. If  $\mathcal{A}$  admits sequential limits, then we have a left exact functor  $\varprojlim: \text{Fun}(\mathbb{N}^{\text{op}}, \mathcal{A}) \rightarrow \mathcal{A}$ . Given a topos  $T$ , we have a pair of adjoint functors

$$\pi^*: \text{Mod}(T, \Lambda) \rightarrow \text{Mod}(T^{\mathbb{N}}, \Lambda_{\bullet}); \quad \pi_*: \text{Mod}(T^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow \text{Mod}(T, \Lambda)$$

induced by the morphism  $\pi: (\mathbb{N}, \Lambda_{\bullet}) \rightarrow (*, \Lambda)$ . Then  $\pi_* = \varprojlim \circ \nu$ , where  $\nu: \text{Mod}(T^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow \text{Fun}(\mathbb{N}^{\text{op}}, \text{Mod}(T, \Lambda))$  is the obvious forgetful functor, which is exact.

**Lemma 2.1.7.** *Let  $F_{\bullet}$  be a module in  $\text{Mod}(T^{\mathbb{N}}, \Lambda_{\bullet})$  such that  $\nu F_{\bullet}$  is essentially null. Then  $R^n \pi_* F_{\bullet} = 0$  for all  $n \geq 0$ .*

*Proof.* Note that  $R^n \pi_* F_{\bullet}$  is the sheaf associated to the presheaf  $(U \mapsto H^n(U^{\mathbb{N}}, F_{\bullet}))$ , where  $U$  runs over objects of  $T$ . Let  $a: (U^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow (*^{\mathbb{N}}, \Lambda_{\bullet})$  be the morphism of ringed topoi. Since  $R^q a_* F_{\bullet}$  is essentially null for all  $q$ , we have  $R\Gamma(U^{\mathbb{N}}, F_{\bullet}) \simeq R\varprojlim R a_* F_{\bullet} = 0$ .  $\square$

Let  $\mathcal{D}_0(X, \Lambda_{\bullet})$  be the full subcategory of  $\mathcal{D}(X, \Lambda_{\bullet})$  spanned by complexes whose cohomology sheaves are all essentially null. Let  $\mathcal{D}_0^{(+)}(X, \Lambda_{\bullet}) = \mathcal{D}^{(+)}(X, \Lambda_{\bullet}) \cap \mathcal{D}_0(X, \Lambda_{\bullet})$ . Both are stable subcategories.

**Lemma 2.1.8.** *For  $\mathcal{K} \in \mathcal{D}_0^{(+)}(X, \Lambda_{\bullet})$ , we have  $R\pi_* \mathcal{K} = 0$ .*

*Proof.* We may assume that  $X$  is a scheme and  $\mathcal{K} \in \mathcal{D}_0^{+}(X, \Lambda_{\bullet})$ . Then the statement follows from Lemma 2.1.7.  $\square$

**Lemma 2.1.9.** *The cohomological amplitude of  $\pi^*$  is contained in  $[-1, 0]$ .*

*Proof.* By assumption,  $\mathfrak{m} = (\lambda)$ , where  $\lambda \in \Lambda$  is not a zero divisor. Since  $\Lambda \xrightarrow{\times \lambda^{n+1}} \Lambda$  is a resolution of  $\Lambda_n$ , the tor-dimension of the  $\Lambda$ -module  $\Lambda_n$  is  $\leq 1$ .  $\square$



**Definition 2.1.10.** A complex  $\mathcal{K} \in \mathcal{D}(X, \Lambda_\bullet)$  is called *essentially normalized* if the cofiber of the adjunction map  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \rightarrow \mathcal{K}$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . We denote by  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$  the full subcategory of  $\mathcal{D}(X, \Lambda_\bullet)$  spanned by essentially normalized complexes, which is a stable subcategory.

**Lemma 2.1.11.** *The image of the functor  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_*|_{\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)}$  is contained in  $\mathcal{D}_n(X, \Lambda_\bullet)$ . Moreover, the induced functor  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet) \rightarrow \mathcal{D}_n(X, \Lambda_\bullet)$  is right adjoint to the inclusion  $\mathcal{D}_n(X, \Lambda_\bullet) \subseteq \mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ .*

*Proof.* For the first assertion, we need to show that  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \rightarrow \mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K}$  is an equivalence for  $\mathcal{K} \in \mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ . By definition, the cofiber of  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \rightarrow \mathcal{K}$  is contained in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . The assertion then follows from Lemma 2.1.8.

For the second assertion, we need to show that the natural transformation  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_* \rightarrow \text{id}$  induces a homotopy equivalence (i.e. an equivalence in  $\mathcal{H}$ )

$$\text{Map}_{\mathcal{D}_n(X, \Lambda_\bullet)}(\mathcal{K}, \mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{L}) \rightarrow \text{Map}_{\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)}(\mathcal{K}, \mathcal{L}),$$

for every object  $\mathcal{K}$  (resp.  $\mathcal{L}$ ) of  $\mathcal{D}_n(X, \Lambda_\bullet)$  (resp.  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ ). By definition, the cofiber  $\mathcal{L}'$  of  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{L} \rightarrow \mathcal{L}$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ , and  $\mathcal{K}$  is equivalent to  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K}$ . Therefore, the assertion follows from the fact that

$$\text{Map}_{\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)}(\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K}, \mathcal{L}') \simeq \text{Map}_{\mathcal{D}(X, \Lambda)}(\mathbf{R}\pi_*\mathcal{K}, \mathbf{R}\pi_*\mathcal{L}') \simeq \{*\}.$$

Here in the second equivalence we have used the fact that  $\mathbf{R}\pi_*\mathcal{L}' = 0$ , which follows from Lemma 2.1.8.  $\square$

**Lemma 2.1.12.** *Let  $\mathcal{K} \in \underline{\mathcal{D}}(X, \Lambda_\bullet) \cap \mathcal{D}^{(-)}(X, \Lambda_\bullet)$ ,  $\mathcal{L} \in \mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . Then  $\mathbf{Hom}(\mathcal{K}, \mathcal{L}) \in \mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$  and  $\mathbf{Hom}(\mathcal{K}, \mathcal{L}) = 0$ .*

*Proof.* The proof is similar to [26, 4.19].  $\square$

**Lemma 2.1.13.** *For every  $\mathcal{K} \in \mathcal{D}_n(X, \Lambda_\bullet)$  and  $n \in \mathbb{Z}$ ,  $\tau^{\geq n}\mathcal{K}$  is in  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ . Moreover, the fiber of the adjunction map  $\mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq n}\mathcal{K} \rightarrow \tau^{\geq n}\mathcal{K}$  is concentrated in degree  $n - 1$ .*

*Proof.* This is essentially proved in [26, 4.14]. Let us recall the arguments. The fiber of the map  $a: \mathbf{L}\pi^*\tau^{\geq n}\mathbf{R}\pi_*\mathcal{K} \rightarrow \tau^{\geq n}\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \simeq \tau^{\geq n}\mathcal{K}$  is concentrated in degree  $n - 1$  and belongs to  $\mathcal{D}_0(X, \Lambda_\bullet)$ . Consider the diagram

$$\begin{array}{ccc} \mathbf{L}\pi^*\mathbf{R}\pi_*\mathbf{L}\pi^*\tau^{\geq n}\mathbf{R}\pi_*\mathcal{K} & \xrightarrow{b} & \mathbf{L}\pi^*\tau^{\geq n}\mathbf{R}\pi_*\mathcal{K} \\ \mathbf{L}\pi^*\mathbf{R}\pi_*a \downarrow & & \downarrow a \\ \mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq n}\mathcal{K} & \xrightarrow{c} & \tau^{\geq n}\mathcal{K}. \end{array}$$

By Lemma 2.1.7,  $\mathbf{L}\pi^*\mathbf{R}\pi_*a$  is an equivalence. By Proposition 2.1.3,  $b$  is an equivalence. Therefore, the fiber of  $c$  is equivalent to the fiber of  $a$ .  $\square$

We denote by  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)$  the full subcategory of  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$  spanned by complexes that are concentrated at degree 0, and  $\text{Mod}_0(X, \Lambda_\bullet)$  the full subcategory of  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)$  spanned by essentially null modules. Then  $\text{Mod}_0(X, \Lambda_\bullet)$  is closed under sub-objects, quotients and extensions.

**Proposition 2.1.14.** *For  $n \in \mathbb{Z}$ , let  $\mathcal{D}_{\text{en}}^{\leq n}(X, \Lambda_\bullet) = \mathcal{D}^{\leq n}(X, \Lambda_\bullet) \cap \mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$  and  $\mathcal{D}_{\text{en}}^{\geq n}(X, \Lambda_\bullet) = \mathcal{D}^{\geq n}(X, \Lambda_\bullet) \cap \mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ . Then  $(\mathcal{D}_{\text{en}}^{\leq 0}(X, \Lambda_\bullet), \mathcal{D}_{\text{en}}^{\geq 0}(X, \Lambda_\bullet))$  defines a  $t$ -structure on  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$  with heart  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)$ , and  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)$  is (equivalent to the nerve of) a full subcategory of  $\text{Mod}(X, \Lambda_\bullet)$ , closed under kernels, cokernels and extensions.*

*Proof.* We only need to show that  $\tau^{\leq 0}$  and  $\tau^{\geq 0}$  preserve the full subcategory  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ . Since  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$  is a stable full subcategory, we only need to prove this for  $\tau^{\geq 0}$ , that is, the cofiber of



the adjunction map  $\mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq 0}\mathcal{K} \rightarrow \tau^{\geq 0}\mathcal{K}$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$  for every object  $\mathcal{K}$  of  $\mathcal{D}_{\text{en}}(X, \Lambda_\bullet)$ . Consider the diagram

$$\begin{array}{ccc} \mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq 0}\mathcal{K} & \xrightarrow{b} & \tau^{\geq 0}\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \\ \mathbf{L}\pi^*\mathbf{R}\pi_*a \downarrow & & \downarrow a \\ \mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq 0}\mathcal{K} & \xrightarrow{c} & \tau^{\geq 0}\mathcal{K}. \end{array}$$

By definition, the cofiber of  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \rightarrow \mathcal{K}$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ , so that the cofiber of  $a$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . It follows that  $\mathbf{L}\pi^*\mathbf{R}\pi_*a$  is an equivalence, by Lemma 2.1.8. By Lemma 2.1.11,  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \in \mathcal{D}_{\text{n}}(X, \Lambda_\bullet)$ . Thus, by the first part of Lemma 2.1.13, the cofiber of  $b$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . Therefore, by the octahedral axiom, the cofiber of  $c$  is in  $\mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$  as well.  $\square$

**Corollary 2.1.15.** *The essential image of  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_* | \mathcal{D}_{\text{en}}^{\geq n}(X, \Lambda_\bullet)$  is right perpendicular to the full subcategory  $\mathcal{D}_{\text{n}}(X, \Lambda_\bullet) \cap \mathcal{D}^{< n}(X, \Lambda_\bullet)$  of  $\mathcal{D}_{\text{n}}(X, \Lambda_\bullet)$ .*

**2.2. Constructible adic complexes.** Let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}$  such that  $\Lambda/\mathfrak{m}^{n+1}$  is Noetherian for all  $n$ . For a higher Artin stack  $X$ , we define

$$\underline{\mathcal{D}}_{\text{cons}}(X, \Lambda_\bullet) = \underline{\mathcal{D}}(X, \Lambda_\bullet) \cap \mathcal{D}_{\text{cons}}(X, \Lambda_\bullet),$$

$$\underline{\mathcal{D}}_{\text{cons}}^{(+)}(X, \Lambda_\bullet) = \underline{\mathcal{D}}(X, \Lambda_\bullet) \cap \mathcal{D}_{\text{cons}}^{(+)}(X, \Lambda_\bullet), \quad \underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda_\bullet) = \underline{\mathcal{D}}(X, \Lambda_\bullet) \cap \mathcal{D}_{\text{cons}}^{(-)}(X, \Lambda_\bullet).$$

The following is an immediate consequence of the definitions and [18, 6.2.3].

**Lemma 2.2.1.** *Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks. Then  $f^*$  and  $- \otimes -$  restrict to the following:*

$$\mathbf{1L}': \underline{f}^*: \underline{\mathcal{D}}_{\text{cons}}(X, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}(Y, \Lambda_\bullet);$$

$$\mathbf{3L}': - \otimes_X -: \underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda_\bullet) \times \underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda_\bullet).$$

In particular,  $\underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda)^\otimes$  [20, 2.2.1] is a symmetric monoidal category.

As in [18, 6.2], to state the results for the other operations, we work in a relative setting. Let  $S$  be an  $\mathbf{L}$ -coprime higher Artin stack. Assume that there exists an atlas  $S \rightarrow \mathbf{S}$ , where  $\mathbf{S}$  is either a quasi-excellent scheme or a regular scheme of dimension  $\leq 1$ . Combining [18, 6.2.4] and Propositions 2.1.4, 2.1.5, 2.1.6, we have the following.

**Proposition 2.2.2.** *Let  $f: Y \rightarrow X$  be a morphism of  $\text{Chp}_{\text{lft}/S}^{\text{Ar}}$ . Assume  $(\Lambda, \mathfrak{m}) \in \mathcal{P}\text{Ring}_{\mathbf{L}\text{-tor}}$ . Then  $f_*$ ,  $f_!$ ,  $f^!$ ,  $\mathbf{Hom}$  restrict to the following:*

$$\mathbf{1R}': \underline{f}_* : \underline{\mathcal{D}}_{\text{cons}}^{(+)}(Y, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}^{(+)}(X, \Lambda_\bullet), \text{ if } f \text{ is quasi-compact and quasi-separated, and } \underline{f}_* : \underline{\mathcal{D}}_{\text{cons}}(Y, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}(X, \Lambda_\bullet) \text{ if } \mathbf{S} \text{ is locally finite-dimensional and } f \text{ is quasi-compact, quasi-separated, and 0-Artin};$$

$$\mathbf{2L}': \underline{f}_! : \underline{\mathcal{D}}_{\text{cons}}^{(-)}(Y, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}^{(-)}(X, \Lambda_\bullet), \text{ if } f \text{ is quasi-compact and quasi-separated};$$

$$\mathbf{2R}': \underline{f}^! : \underline{\mathcal{D}}_{\text{cons}}^{(+)}(X, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}^{(+)}(Y, \Lambda_\bullet), \text{ and, if } \mathbf{S} \text{ is locally finite-dimensional, } \underline{f}^! : \underline{\mathcal{D}}_{\text{cons}}(X, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}(Y, \Lambda_\bullet);$$

$$\mathbf{3R}': \underline{\mathbf{Hom}}_X(-, -) : \underline{\mathcal{D}}_{\text{cons}}^{(\text{b})}(X, \Lambda_\bullet)^{\text{op}} \times \underline{\mathcal{D}}_{\text{cons}}^{(+)}(X, \Lambda_\bullet) \rightarrow \underline{\mathcal{D}}_{\text{cons}}^{(+)}(X, \Lambda_\bullet).$$

**2.3. Finiteness conditions and the usual  $t$ -structure.** Let  $X$  be a higher Artin stack and  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}$ . Recall that  $\mathcal{D}_{\text{n}}(X, \Lambda_\bullet) \subseteq \underline{\mathcal{D}}(X, \Lambda_\bullet)$ .

*Definition 2.3.1.* The pair  $(X, (\Lambda, \mathfrak{m}))$  is said to be *admissible* if  $\underline{\mathcal{D}}(X, \Lambda_\bullet) \subseteq \mathcal{D}_{\text{n}}(X, \Lambda_\bullet)$  (so that  $\underline{\mathcal{D}}(X, \Lambda_\bullet) = \mathcal{D}_{\text{n}}(X, \Lambda_\bullet)$ ), that is, for every  $\mathcal{K} \in \underline{\mathcal{D}}(X, \Lambda_\bullet)$ , the adjunction map  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \rightarrow \mathcal{K}$  is an equivalence.

**Proposition 2.3.2.** *Let  $(X, (\Lambda, \mathfrak{m}))$  be an admissible pair. Then  $R_X | \mathcal{D}_{\text{en}}(X, \Lambda_\bullet) \simeq \mathbf{L}\pi^* \circ \mathbf{R}\pi_*$ , and  $\underline{\mathcal{D}}^{\geq n}(X, \Lambda_\bullet)$  is the essential image of  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_* | \mathcal{D}_{\text{en}}^{\geq n}(X, \Lambda_\bullet)$ .*

*Proof.* The first assertion follows from Lemma 2.1.11. We denote by  $\mathcal{D}'$  the essential image of  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_* | \mathcal{D}_{\text{en}}^{\geq n}(X, \Lambda_\bullet)$  in the second assertion. Then  $\mathcal{D}' \subseteq \mathcal{D}^{\geq n}(X, \Lambda_\bullet)$  by Corollary 2.1.15. Moreover, since  $\mathbf{L}\pi^*: \mathcal{D}(X, \Lambda) \rightarrow \mathcal{D}(X, \Lambda_\bullet)$  is left  $t$ -exact,  $\mathbf{R}\pi_* | \mathcal{D}(X, \Lambda_\bullet): \mathcal{D}(X, \Lambda_\bullet) \rightarrow \mathcal{D}(X, \Lambda)$  is right  $t$ -exact. Thus, for  $\mathcal{K} \in \mathcal{D}^{\geq n}(X, \Lambda_\bullet)$ , the fiber of  $\mathcal{K} \rightarrow \tau^{\geq n}\mathcal{K}$  (where  $\tau^{\geq n}$  is the truncation functor in  $\mathcal{D}(X, \Lambda_\bullet)$ ), equivalent to the fiber of  $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \simeq \mathbf{L}\pi^*\tau^{\geq n}\mathbf{R}\pi_*\mathcal{K} \rightarrow \tau^{\geq n}\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K}$ , is concentrated in degree  $n-1$  and belongs to  $\mathcal{D}_0(X, \Lambda_\bullet)$ . Therefore,  $\mathcal{K} \simeq \mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} \simeq \mathbf{L}\pi^*\mathbf{R}\pi_*\tau^{\geq n}\mathcal{K}$ , which belongs to  $\mathcal{D}'$  by Lemma 2.1.13.  $\square$

*Remark 2.3.3.* Let  $\text{Mod}'_{\text{en}}(X, \Lambda_\bullet)$  be the full subcategory of  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)$  spanned by complexes (modules)  $\mathcal{K}$  such that  $\mathbf{H}^i\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{K} = 0$  for  $i > 0$ , which is an exact category. Then the projection functor  $\mathcal{D}^\heartsuit(X, \Lambda_\bullet) \rightarrow \text{Mod}'_{\text{en}}(X, \Lambda_\bullet)/\text{Mod}_0(X, \Lambda_\bullet)$  from the heart of the usual  $t$ -structure to the full subcategory of  $\text{Mod}_{\text{en}}(X, \Lambda_\bullet)/\text{Mod}_0(X, \Lambda_\bullet)$  spanned by the image of  $\text{Mod}'_{\text{en}}(X, \Lambda_\bullet)$  is an equivalence of categories. In fact,  $\mathbf{L}\pi^* \circ \mathbf{R}\pi_* | \text{Mod}'_{\text{en}}(X, \Lambda_\bullet)$  induces a quasi-inverse.

**Proposition 2.3.4.** *Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks.*

- (1) *Let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}$  such that  $(X, (\Lambda, \mathfrak{m}))$  is admissible. Then  $\underline{f}^*: \mathcal{D}(X, \Lambda_\bullet) \rightarrow \mathcal{D}(Y, \Lambda_\bullet)$  is  $t$ -exact with respect to the usual  $t$ -structures.*
- (2) *Assume  $f$  is locally of finite type (resp. locally of finite type, resp. a locally quasi-finite morphism of Deligne–Mumford stacks). Let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}_{\text{L-tor}}$  (resp.  $\mathcal{P}\text{Ring}_{\text{tor}}$ , resp.  $\mathcal{P}\text{Ring}$ ) such that  $(Y, (\Lambda, \mathfrak{m}))$  is admissible. Then  $\underline{f}_!: \mathcal{D}(Y, \Lambda_\bullet) \rightarrow \mathcal{D}(X, \Lambda_\bullet)$  is right  $t$ -exact with respect to the usual  $t$ -structures.*

*Proof.* (1) We only need to show the right  $t$ -exactness of  $\underline{f}^*$ . Let  $\mathcal{K} \in \mathcal{D}^{\geq n}(X, \Lambda_\bullet)$ ,  $\mathcal{L} \in \mathcal{D}^{\leq n-1}(Y, \Lambda_\bullet)$ . Consider the fiber sequence  $\tau^{\leq n-1}\mathcal{K} \rightarrow \mathcal{K} \rightarrow \tau^{\geq n}\mathcal{K}$  in  $\mathcal{D}(X, \Lambda_\bullet)$ . It induces a fiber sequence  $f^*\tau^{\leq n-1}\mathcal{K} \rightarrow \underline{f}^*\mathcal{K} \rightarrow f^*\tau^{\geq n}\mathcal{K}$  in  $\mathcal{D}(Y, \Lambda_\bullet)$ . By assumption and Proposition 2.3.2,  $\tau^{\leq n-1}\mathcal{K} \in \mathcal{D}_0^{(+)}(X, \Lambda_\bullet)$ . Thus  $f^*\tau^{\leq n-1}\mathcal{K} \in \mathcal{D}_0^{(+)}(Y, \Lambda_\bullet)$ , and  $\text{Hom}(\mathcal{L}, f^*\tau^{\leq n-1}\mathcal{K}) = 0$  by 2.1.12. It follows that  $\text{Hom}(\mathcal{L}, \underline{f}^*\mathcal{K}) = 0$ . Therefore,  $\underline{f}^*\mathcal{K} \in \mathcal{D}^{\geq n}(Y, \Lambda_\bullet)$ .  
(2) Similar to (1).  $\square$

Let  $f: Y \rightarrow X$  be a smooth surjective morphism in  $\text{Chp}_{\text{L}}^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ ),  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}_{\text{L-tor}}$  (resp.  $\mathcal{P}\text{Ring}$ ). By Poincaré duality, if  $(Y, (\Lambda, \mathfrak{m}))$  is admissible, then  $(X, (\Lambda, \mathfrak{m}))$  is locally admissible. This applies in particular to the case where  $Y$  is an algebraic space. In this case, admissibility is related to the following finiteness condition on cohomological dimension.

*Definition 2.3.5.* Let  $X$  be a higher Artin (resp. Deligne–Mumford) stack,  $R$  be a ring. We say  $X$  is *locally  $R$ -bounded*, if there exists an atlas (resp. étale atlas)  $\coprod_{i \in I} X_i \rightarrow X$  with  $X_i$  algebraic spaces such that for every  $i \in I$ , and every scheme  $U$  étale and of finite presentation over  $X_i$ ,

$$\text{cd}_R(U) := \max\{n \mid \mathbf{H}^n(U, F) \neq 0 \text{ for some } F \in \text{Mod}(U, R)\} < \infty.$$

**Proposition 2.3.6.** *Let  $X$  be an algebraic space,  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}$ . Consider the following conditions:*

- (1) *The pair  $(X, (\Lambda, \mathfrak{m}))$  is admissible.*
- (2) *For every  $\mathcal{K} \in \mathcal{D}(X, \Lambda_\bullet)$ ,  $\mathbf{R}\pi_*(F_\bullet \otimes_{\Lambda_\bullet} \mathcal{K}) = 0$ , where  $F_\bullet \in \text{Mod}(X, \Lambda_\bullet) \simeq \mathcal{D}^\heartsuit(X, \Lambda_\bullet)$  is*

$$\cdots \xrightarrow{0} \Lambda/\mathfrak{m} \xrightarrow{0} \cdots \xrightarrow{0} \Lambda/\mathfrak{m}.$$

- (3)  *$\mathbf{R}\pi_*\mathcal{K} = 0$  for every  $\mathcal{K} \in \mathcal{D}_0(X, \Lambda_\bullet)$ .*
- (4) *There exists an étale cover  $\coprod_{i \in I} X_i \rightarrow X$  by algebraic spaces such that, for every  $i \in I$ , the cohomological dimension of  $\pi_*: \text{Mod}(X_{i, \text{ét}}^{\text{N}}, \Lambda_\bullet) \rightarrow \text{Mod}(X_{i, \text{ét}}, \Lambda)$  is finite.*
- (5) *The algebraic space  $X$  is locally  $(\Lambda/\mathfrak{m})$ -bounded.*

We have (5)  $\Rightarrow$  (4)  $\Rightarrow$  (3)  $\Rightarrow$  (2)  $\Leftrightarrow$  (1).

*Proof.* (5)  $\Rightarrow$  (4): By étale base change, we can assume that for every scheme  $U$  étale and of finite type over  $X$ ,  $\mathrm{cd}_{\Lambda/\mathfrak{m}}(U) = N < \infty$ . Since for  $n \in \mathbb{N}$ , every  $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$ -module is a successive extension of  $\Lambda/\mathfrak{m}$ -modules, we have  $\mathrm{cd}_{\Lambda_n}(U) = N$ . For a sheaf  $F_\bullet \in \mathrm{Mod}(X_{\mathrm{\acute{e}t}}^{\mathbb{N}}, \Lambda_\bullet)$ ,  $R^i \pi_* F_\bullet$  is the sheaf associated to the presheaf  $U \mapsto H^i(U_{\mathrm{\acute{e}t}}^{\mathbb{N}}, F_\bullet)$ . Thus, from the exact sequence

$$0 \longrightarrow R^1 \varprojlim_n H^{i-1}(U_{\mathrm{\acute{e}t}}, F_n) \longrightarrow H^i(U_{\mathrm{\acute{e}t}}^{\mathbb{N}}, F_\bullet) \longrightarrow \varprojlim_n H^i(U_{\mathrm{\acute{e}t}}, F_n) \longrightarrow 0,$$

we know that  $R^i \pi_* F_\bullet = 0$  for  $i > N + 1$ .

(4)  $\Rightarrow$  (3): We can assume  $X$  to be quasi-compact. Then this follows from Lemma 2.1.8 and the following standard observation. Let  $f: \mathcal{B} \rightarrow \mathcal{A}$  be a left exact additive functor of Grothendieck Abelian categories, such that  $R^i f = 0$  for  $i > d$  where  $d$  is a non-negative integer. Then  $Rf$  sends  $D^{\leq n}(\mathcal{B})$  to  $D^{\leq n+d}(\mathcal{A})$ . In fact, let  $X$  be an element of  $D^{\leq n}(\mathcal{B})$ . By [12, 14.3.4], we can compute  $RfX$  by  $fY$ , where  $Y$  is *any* resolution of  $X$  with  $f$ -acyclic components. We can take  $Y$  to be  $\tau^{\leq n+d+1}$  of a homotopically injective resolution with injective components (fibrant replacement) of  $X$ . Then  $Y$  belongs to  $C^{\leq n+d+1}(\mathcal{B})$ . This shows that  $Rf$  sends  $D^{\leq n}(\mathcal{B})$  to  $D^{\leq n+d+1}(\mathcal{A})$ . It follows that  $Rf$  sends  $D^{\leq n}(\mathcal{B})$  to  $D^{\leq n+d}(\mathcal{A})$  by truncation.

(3)  $\Rightarrow$  (2): In fact, for every  $\mathcal{K} \in \mathcal{D}(X, \Lambda_\bullet)$ ,  $F_\bullet \otimes_{\Lambda_\bullet} \mathcal{K}$  belongs to  $\mathcal{D}_0(X, \Lambda_\bullet)$ .

(2)  $\Leftrightarrow$  (1): Let  $\mathcal{K} \in \underline{\mathcal{D}}(X, \Lambda_\bullet)$ . We need to show that  $R\pi_*(F_\bullet \otimes_{\Lambda_\bullet} \mathcal{K}) = 0$  if and only if the adjunction map  $L\pi^* R\pi_* \mathcal{K} \rightarrow \mathcal{K}$  is an equivalence. Since  $\prod_{n \in \mathbb{N}} e_n^*$  is conservative, the latter is equivalent to the condition that the morphism  $\epsilon: \Lambda_n \otimes_{\Lambda}^L R\pi_* \mathcal{K} \rightarrow \mathcal{K}_n := e_n^* \mathcal{K}$  is an isomorphism in the derived category  $D(X, \Lambda_n)$  for all  $n \in \mathbb{N}$ . The morphism  $\epsilon$  can be decomposed as

$$\begin{aligned} \Lambda_n \otimes_{\Lambda}^L R\pi_* \mathcal{K} &\xrightarrow{\alpha} R\pi_*(L\pi^* \Lambda_n \otimes_{\Lambda_\bullet}^L \mathcal{K}) \xrightarrow{\beta} R\pi_*(\pi^* \Lambda_n \otimes_{\Lambda_\bullet}^L \mathcal{K}) \\ &\xrightarrow{\gamma} R(\pi_{\geq n})_*(\pi_{\geq n}^* \Lambda_n \otimes_{\Lambda_{\bullet, \geq n}}^L \mathcal{K}_{\geq n}) \xrightarrow{\delta} R(\pi_{\geq n})_*(e_{n*} \mathcal{K}_n) \simeq \mathcal{K}_n, \end{aligned}$$

where  $\pi_{\geq n}: (\mathbb{N}_{\geq n}, \Lambda_{\bullet, \geq n}) \rightarrow (*, \Lambda)$ ,  $e_n: (\{n\}, \Lambda) \rightarrow (\mathbb{N}_{\geq n}, \Lambda_{\bullet, \geq n})$ . Here  $\mathbb{N}_{\geq n} \subseteq \mathbb{N}$  is the full subcategory spanned by integers  $\geq n$ .

By assumption,  $\mathfrak{m}$  is generated by an element  $\lambda$  that is not a zero divisor. Thus we have a finite free resolution  $[\Lambda \xrightarrow{\times \lambda^{n+1}} \Lambda]$  of  $\Lambda_n$  as an  $\Lambda$ -module. Therefore,  $L\pi^* \Lambda_n$  is represented by the complex of  $\Lambda_\bullet$ -modules  $[\Lambda_\bullet \xrightarrow{\times \lambda^{n+1}} \Lambda_\bullet]$  (in degrees  $-1$  and  $0$ ). This implies that  $L\pi^* \Lambda_n \otimes_{\Lambda_\bullet}^L \mathcal{K}$  is represented by the mapping cone of  $\mathcal{K} \xrightarrow{\times \lambda^{n+1}} \mathcal{K}$ , which is a fibrant object. Then  $\Lambda_n \otimes_{\Lambda}^L R\pi_* \mathcal{K}$  and  $R\pi_*(L\pi^* \Lambda_n \otimes_{\Lambda_\bullet}^L \mathcal{K})$  are both represented by  $\pi_* \mathcal{K}' \xrightarrow{\times \lambda^{n+1}} \pi_* \mathcal{K}'$ , where  $\mathcal{K}'$  is a fibrant replacement of  $\mathcal{K}$ , and  $\alpha$  is represented by the identity.

Consider the diagram

$$\begin{array}{ccc} (\mathbb{N}_{\geq n}, \Lambda_{\bullet, \geq n}) & \xrightarrow{j} & (\mathbb{N}, \Lambda_\bullet) \\ \downarrow & & \downarrow \\ (\mathbb{N}_{\geq n}, \lambda_{\geq n}) & \xrightarrow{j'} & (\mathbb{N}, \lambda) \\ & \searrow \pi'_{\geq n} & \downarrow \pi' \\ & & \Lambda, \end{array}$$

where  $\lambda$  is the constant ring with value  $\Lambda$ . By the cofinality of  $\mathbb{N}_{\geq n}$  in  $\mathbb{N}$ , the natural transformation  $\pi'_* \rightarrow (\pi'_{\geq n})_* \circ j'^*$  is an isomorphism. Since  $j'_*$  admits an *exact* left adjoint, it follows that  $R\pi'_* \rightarrow R(\pi'_{\geq n})_* \circ j'^*$  is an isomorphism. Thus the natural transformation  $R\pi_* \rightarrow R\pi_{\geq n} \circ j^*$  is an isomorphism. Therefore,  $\gamma$  is an isomorphism.

The morphism  $\delta$  is induced by the morphism  $\pi_{\geq n}^* \Lambda_n \otimes_{\Lambda_{\bullet, \geq n}}^{\mathbf{L}} \mathcal{K}_{\geq n} \rightarrow e_{n*} e_n^* (\pi_{\geq n}^* \Lambda_n \otimes_{\Lambda_{\bullet, \geq n}}^{\mathbf{L}} \mathcal{K}_{\geq n}) \simeq e_{n*} \mathcal{K}_n$ , which is an isomorphism since  $\mathcal{K}$  is adic. Therefore,  $\epsilon$  is an isomorphism if and only if  $\beta$  is an isomorphism.

By the above resolution of  $\Lambda$ , the cone of  $\mathbf{L}\pi^* \Lambda_n \rightarrow \Lambda_n$  is  $G_{\bullet}^n[-2]$ , where  $G_m^n = \Lambda/\mathfrak{m}^{\min(m, n)+1}$  and the transition maps are multiplication by  $\lambda$ , so that  $G_{\bullet}^0 = F_{\bullet}$ . Thus, if  $\beta$  is an isomorphism for  $n = 0$ , then  $\mathbf{R}\pi_*(F_{\bullet} \otimes_{\Lambda_{\bullet}} \mathcal{K}) = 0$ . For  $n \geq 1$ ,  $G_{\bullet}^n$  is an extension of  $F_{\bullet}$  by  $G_{\bullet+1}^{n-1}$ . Thus, if  $\mathbf{R}\pi_*(F_{\bullet} \otimes_{\Lambda_{\bullet}} \mathcal{K}) = 0$ , then, by the above,  $\beta$  is an isomorphism for all  $n \in \mathbb{N}$ .  $\square$

**2.4. Unbounded constructible adic complexes.** Let  $(\Lambda, \mathfrak{m})$  be an object of  $\mathcal{P}\text{Ring}_{\mathbf{L}\text{-tor}}$  such that  $\Lambda/\mathfrak{m}^{n+1}$  is Noetherian for all  $n$ . Let  $\mathbf{S}$  be an  $\mathbf{L}$ -coprime higher Artin stack. Assume that there exists an atlas  $S \rightarrow \mathbf{S}$ , where  $S$  is either a quasi-excellent scheme or a regular scheme of dimension  $\leq 1$ .

**Proposition 2.4.1.** *Let  $f: Y \rightarrow X$  be a morphism of  $\text{Chp}_{\text{ift}/S}^{\text{Ar}}$ . Then  $f^!$  and  $\mathbf{Hom}$  restrict to the following:*

- 2R':**  $f^!: \mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet}) \rightarrow \mathcal{D}_{\text{cons}}(Y, \Lambda_{\bullet})$ , if  $(X, (\Lambda, \mathfrak{m}))$  and  $(Y, (\Lambda, \mathfrak{m}))$  are admissible.
- 3R':**  $\mathbf{Hom}_X(-, -): \mathcal{D}_{\text{cons}}^{(-)}(X, \Lambda_{\bullet})^{\text{op}} \times \mathcal{D}_{\text{cons}}^{(+)}(X, \Lambda_{\bullet}) \rightarrow \mathcal{D}_{\text{cons}}^{(+)}(X, \Lambda_{\bullet})$ , if  $(X, (\Lambda, \mathfrak{m}))$  is admissible.

Let  $X$  be a scheme in  $\text{Sch}^{\text{qc.sep}}$ . Recall that a complex  $\mathcal{K} \in \mathcal{D}(X, \Lambda_{\bullet})$  is a  $\lambda$ -complex [14, 3.0.6] if  $H^n \mathcal{K}$  is constructible and almost adic. In particular,  $\mathcal{K} \in \mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet})$ . The proofs of the following statements are similar to [26].

**Lemma 2.4.2.** *Let  $X$  be a scheme in  $\text{Sch}^{\text{qc.sep}}$  such that Condition (3) in Proposition 2.3.6 holds for the pair  $(X, (\Lambda, \mathfrak{m}))$ . Let  $\mathcal{D}'_{\text{cons}}(X, \Lambda_{\bullet})$  be the full subcategory of  $\mathcal{D}_{\text{en}}(X, \Lambda_{\bullet})$  spanned by  $\lambda$ -complexes. We have*

- (1)  $\mathcal{D}'_{\text{cons}}(X, \Lambda_{\bullet})$  is closed under the truncation functors  $\tau^{\geq n}$  and  $\tau^{\leq n}$ .
- (2) The essential image of  $\mathbf{L}\pi^* \mathbf{R}\pi_* \mathcal{D}'_{\text{cons}}(X, \Lambda_{\bullet})$  coincides with

$$\mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet}) = \mathcal{D}(X, \Lambda_{\bullet}) \cap \mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet}).$$

**Proposition 2.4.3.** *Let the assumptions be as in the above lemma. Put*

$$\mathcal{D}_{\text{cons}}^{\leq n}(X, \Lambda_{\bullet}) = \mathcal{D}^{\leq n}(X, \Lambda_{\bullet}) \cap \mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet}).$$

*Then the right perpendicular full subcategory  $\mathcal{D}_{\text{cons}}^{\geq n}(X, \Lambda_{\bullet})$  of  $\mathcal{D}_{\text{cons}}^{\leq n-1}(X, \Lambda_{\bullet})^{\perp}$  in  $\mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet})$  is the essential image of  $\mathbf{L}\pi^* \mathbf{R}\pi_*(\mathcal{D}'_{\text{cons}}(X, \Lambda_{\bullet}) \cap \mathcal{D}_{\text{en}}^{\geq n}(X, \Lambda_{\bullet}))$ . Moreover, the truncation functors  $\mathcal{D}^{\leq n} \simeq \mathbf{L}\pi^* \circ \mathbf{R}\pi_* \circ \tau^{\leq n}$  and  $\mathcal{D}^{\geq n} \simeq \mathbf{L}\pi^* \circ \mathbf{R}\pi_* \circ \tau^{\geq n}$ .*

**Corollary 2.4.4.** *Let  $X$  be a higher Artin stack that is locally  $(\Lambda/\mathfrak{m})$ -bounded. Then the full subcategory  $\mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet})$  is preserved under the truncation functors  $\mathcal{D}^{\leq n}$  and hence  $\mathcal{D}^{\geq n}$  on  $\mathcal{D}(X, \Lambda_{\bullet})$ .*

**2.5. Compatibility with Laszlo–Olsson.** We prove the compatibility between our adic formalism and Laszlo–Olsson's [14], under their assumptions.

Let  $\mathbf{L} = \{\ell\}$ . Let  $\mathbf{S}$  be an  $\mathbf{L}$ -coprime scheme satisfying that

- (1) It is affine excellent and finite-dimensional;
- (2) For every  $\mathbf{S}$ -scheme  $X$  of finite type, there exists an étale cover  $X' \rightarrow X$  such that  $\text{cd}_{\ell}(Y) < \infty^5$  for every scheme  $Y$  étale and of finite type over  $X'$ ;
- (3) It admits a global dimension function and we fix such a function (see [18, 6.3.1]).

Fix a complete discrete valuation ring  $\Lambda$  with the maximal ideal  $\mathfrak{m}$  and residue characteristic  $\ell$  such that  $\Lambda = \varprojlim_n \Lambda_n$ , where  $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$ , as in [14]. In particular,  $(\Lambda, \mathfrak{m})$  is an object of  $\mathcal{P}\text{Ring}$ . For every stack  $\mathcal{X}$  in  $\text{Chp}_{\text{ift}/S}^{\text{LMB}}$ , the pair  $\mathcal{X}$  is locally  $(\Lambda/\mathfrak{m})$ -bounded.

<sup>5</sup>According to our notation,  $\text{cd}_{\ell}$  is nothing but  $\text{cd}_{\mathbb{F}_{\ell}}$ .

From the definition of  $\underline{\mathcal{D}}_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)$ , which is the full subcategory of  $\mathcal{D}(\mathcal{X}, \Lambda_\bullet)$  spanned by constructible adic complexes, [14, 3.0.10, 3.0.14, 3.0.18], and [18, 5.3.6], we have a canonical equivalence between categories

$$(2.1) \quad \mathbf{h}\underline{\mathcal{D}}_{\text{cons}}(\mathcal{X}, \Lambda_\bullet) \simeq \mathbf{D}_c(\mathcal{X}, \Lambda),$$

where the latter one is defined in [14, 3.0.6].

**Proposition 2.5.1.** *For a morphism  $f: \mathcal{Y} \rightarrow \mathcal{X}$  of finite type in  $\text{Chp}_{\text{ft}/S}^{\text{LMB}}$ , there are natural isomorphisms between functors:*

$$\begin{aligned} \mathbf{h}\underline{f}^* &\simeq \mathbf{L}f^*: \mathbf{D}_c(\mathcal{X}, \Lambda) \rightarrow \mathbf{D}_c(\mathcal{Y}, \Lambda), & \mathbf{h}\underline{f}_* &\simeq \mathbf{R}f_*: \mathbf{D}_c^{(+)}(\mathcal{Y}, \Lambda) \rightarrow \mathbf{D}_c^{(+)}(\mathcal{X}, \Lambda); \\ \mathbf{h}\underline{f}_! &\simeq \mathbf{R}f_!: \mathbf{D}_c^{(-)}(\mathcal{Y}, \Lambda) \rightarrow \mathbf{D}_c^{(-)}(\mathcal{X}, \Lambda), & \mathbf{h}\underline{f}^! &\simeq \mathbf{R}f^!: \mathbf{D}_c(\mathcal{X}, \Lambda) \rightarrow \mathbf{D}_c(\mathcal{Y}, \Lambda); \\ \mathbf{h}(-\underline{\otimes}-) &\simeq (-) \overset{\mathbf{L}}{\otimes} (-): \mathbf{D}_c^{(-)}(\mathcal{X}, \Lambda) \times \mathbf{D}_c^{(-)}(\mathcal{X}, \Lambda) \rightarrow \mathbf{D}_c^{(-)}(\mathcal{X}, \Lambda), \\ \mathbf{h}\underline{\mathbf{Hom}}(-, -) &\simeq \mathcal{R}\mathbf{hom}_\Lambda(-, -): \mathbf{D}_c^{(-)}(\mathcal{X}, \Lambda)^{\text{opp}} \times \mathbf{D}_c^{(+)}(\mathcal{X}, \Lambda) \rightarrow \mathbf{D}_c^{(+)}(\mathcal{X}, \Lambda). \end{aligned}$$

that are compatible with (2.1).

By Lemma 2.2.1 and Proposition 2.2.2, the six operations on the left side in the above proposition do have the correct range.

*Proof.* The isomorphisms for tensor product, internal Hom and  $f^*$  simply follow from the same definitions here and in [14, §§4, 6]. The isomorphism for  $f_*$  follows from the adjunction and that for  $f^*$  ([18, 6.3.2]). The isomorphism for  $f_!$  will follow from the adjunction and that for  $f^!$  which will be proved below.

By the compatibility of dualizing complexes and the isomorphisms for internal Hom, we have natural isomorphisms  $\underline{\mathcal{D}}_{\mathcal{X}} \simeq \mathbf{D}_{\mathcal{X}}$  and  $\underline{\mathcal{D}}_{\mathcal{Y}} \simeq \mathbf{D}_{\mathcal{Y}}$ . Moreover,  $\mathcal{D}(\mathcal{X}, \Lambda_\bullet)_{\text{dual}}$  contains  $\mathcal{D}_{\text{cons}}(\mathcal{X}, \Lambda_\bullet)$ . Therefore, by [14, 9.1], to prove the isomorphism for  $f^!$ , we only need to show that our functors satisfy

$$\mathbf{h}\underline{f}^! \simeq \underline{\mathcal{D}}_{\mathcal{Y}} \circ \mathbf{h}\underline{f}^* \circ \underline{\mathcal{D}}_{\mathcal{X}}.$$

In fact, by the biduality isomorphism, we have

$$\begin{aligned} \mathbf{h}\underline{f}^! &\simeq \mathbf{h}\underline{f}^! \circ \underline{\mathcal{D}}_{\mathcal{X}} \circ \underline{\mathcal{D}}_{\mathcal{X}}(-) \\ &= \mathbf{h}\underline{f}^! \circ \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(\mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}), \Omega_{\mathcal{X}}) \\ &\simeq \mathbf{h}R_{\mathcal{Y}} \circ \mathbf{h}f^! \circ \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(\mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}), \Omega_{\mathcal{X}}) \\ &\simeq \mathbf{h}R_{\mathcal{Y}} \circ \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{Y}}(\mathbf{h}f^* \circ \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}), f^!\Omega_{\mathcal{X}}) \\ &\simeq \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{Y}}(\mathbf{h}\underline{f}^* \circ \mathbf{h}\underline{\mathbf{Hom}}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}), \Omega_{\mathcal{Y}}) \\ &= \underline{\mathcal{D}}_{\mathcal{Y}} \circ \mathbf{h}\underline{f}^* \circ \underline{\mathcal{D}}_{\mathcal{X}}. \end{aligned}$$

□

*Remark 2.5.2.* In view of the above compatibility, we prove all the expected properties of the six operations, in particular the Base Change Theorem, in the adic case of Laszlo–Olsson [14].

### 3. PERVERSE $t$ -STRUCTURES

In §3.1, we define a general notion of perversity, which we call *perversity smooth/étale evaluation* for higher Artin/Deligne–Mumford stacks. We then construct the perverse  $t$ -structure for a perversity evaluation on an Artin stack in §3.2 using descent. In §3.3, we define the adic perverse  $t$ -structure. In both cases, we provide descriptions of the  $t$ -structures in terms of cohomology on stalks as in the classical situation.

### 3.1. Perversity evaluations.

*Definition 3.1.1.* Let  $X$  be a scheme in  $\text{Sch}^{\text{qc.sep}}$ .

- (1) Following [9, §1], a *weak perversity function* on  $X$  is a function  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  such that for every  $n \in \mathbb{Z}$ , the set  $\{x \in |X| \mid p(x) \geq n\}$  is ind-constructible.
- (2) An *admissible perversity function* on  $X$  is a weak perversity function  $p$  such that for every  $x \in |X|$ , there is an open dense subset  $U \subseteq \overline{\{x\}}$  satisfying the condition that for every  $x' \in U$ ,  $p(x') \leq p(x) + 2 \text{codim}(x', x)$ .
- (3) A *codimension perversity function* on  $X$  is a function  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  such that for every immediate étale specialization  $x'$  of  $x$ ,  $p(x') = p(x) + 1$ .

*Remark 3.1.2.*

- (1) A weak perversity function on a locally Noetherian scheme is locally bounded from below.
- (2) An admissible perversity function on a scheme that is locally Noetherian and of finite dimension is locally bounded from above.
- (3) A codimension perversity function on a scheme is *not* necessarily a weak perversity function.
- (4) A codimension perversity function that is also a weak perversity function is an admissible perversity function. If  $X$  is locally Noetherian, then a codimension perversity function is a weak perversity function and hence an admissible perversity function.
- (5) A codimension perversity function is the opposite of a dimension function in the sense of [24, 2.1.8]. If  $X$  is locally Noetherian and admits a dimension function, then  $X$  is universally catenary by [24, 2.2.6]. In this case, immediate étale specializations coincide with immediate Zariski specializations [24, 2.1.4].
- (6) If  $p$  is a weak (resp. admissible, resp. codimension) perversity function on  $X$  and  $d: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  is a locally constant function, then  $p + d$  is a weak (resp. admissible, resp. codimension) perversity function on  $X$ .

*Definition 3.1.3.* A function  $q: \mathbb{N} \rightarrow \mathbb{Z}$  or  $\mathbb{Z} \rightarrow \mathbb{Z}$  is called *admissible* if  $q$  and  $\mathbf{2} - q$  are both increasing, where  $\mathbf{2}(x) = 2x$  and similarly for  $\mathbf{0}$  and  $\mathbf{1}$ , which will be used below.

Let  $f: Y \rightarrow X$  be a smooth morphism of schemes in  $\text{Sch}^{\text{qc.sep}}$ , and let  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$ ,  $q: \mathbb{N} \rightarrow \mathbb{Z}$  be functions. We define the pullback  $f_q^*p: |Y| \rightarrow \mathbb{Z} \cup \{+\infty\}$  by  $(f_q^*p)(y) = p(f(y)) - q(\text{tr.deg}[k(y) : k(f(y))])$  for every point  $y \in |Y|$ . In particular,  $f_{\mathbf{0}}^*p = p \circ f$ .

**Lemma 3.1.4.** *Let  $f: Y \rightarrow X$  be a morphism (resp. étale morphism, resp. étale morphism) of schemes in  $\text{Sch}^{\text{qc.sep}}$ . If  $p$  is a weak (resp. admissible, resp. codimension) perversity function on  $X$ , then  $f_{\mathbf{0}}^*p$  is a weak (resp. admissible, resp. codimension) perversity function on  $Y$ .*

*Proof.* We have  $f_{\mathbf{0}}^*p = p \circ f$ . If  $p$  is a weak perversity function, then

$$\{y \in |Y| \mid f_{\mathbf{0}}^*p(y) \geq n\} = f^{-1}(\{x \in |X| \mid p(x) \geq n\})$$

is ind-constructible by [1, 1.9.5 (vi)]. The other two cases follow from the trivial fact that  $\text{codim}(y', y) = \text{codim}(f(y'), f(y))$  for every specialization  $y'$  of  $y$  on  $Y$ .  $\square$

**Lemma 3.1.5.** *Let  $f: Y \rightarrow X$  be a morphism of locally Noetherian schemes in  $\text{Sch}^{\text{qc.sep}}$ , locally of finite type.*

- (1) *Let  $p$  be a weak perversity function on  $X$ , and let  $q: \mathbb{N} \rightarrow \mathbb{Z}$  be an increasing function. Then  $f_q^*p$  is a weak perversity function on  $Y$ .*
- (2) *Let  $p$  be an admissible perversity function on  $X$ , and let  $q: \mathbb{N} \rightarrow \mathbb{Z}$  be an admissible function. Then  $f_q^*p$  is an admissible perversity function on  $Y$ .*
- (3) *Let  $p$  be a codimension perversity function on  $X$ . Then  $f_{\mathbf{1}}^*p$  is a codimension perversity function on  $Y$ .*



*Proof.* (1) For a locally closed subset  $Z$  of a scheme  $X$ , we endow it with the reduced induced subscheme structure. For every point  $y \in |Y|$ , let  $U_y \subset \overline{\{y\}}$  be a nonempty open subset such that the induced morphism  $f_y: \overline{\{y\}} \rightarrow \overline{\{f(y)\}}$  is flat. Such an open subset exists by [1, 6.9.1]. For  $y' \in U_y$ , we have

$$\delta(y', y) := \text{tr.deg}[k(y) : k(f(y))] - \text{tr.deg}[k(y') : k(f(y'))] = \text{codim}(y', U_y \times_{\overline{f(y)}} \{f(y')\}) \geq 0$$

by [1, 14.3.13] because  $f_y$  is universally open [1, 2.4.6]. Therefore, for  $n \in \mathbb{Z}$ ,

$$\{y \in |Y| \mid f_q^* p(y) \geq n\} = \bigcup_{y \in |Y|} f^{-1} \{x \in |X| \mid p(x) \geq n + q(\text{tr.deg}[k(y) : k(f(y))])\} \cap U_y,$$

is a union of ind-constructible subsets, and hence is itself ind-constructible. In other words,  $f^*p$  is a weak perversity function.

(2) Let  $y \in |Y|$  be a point,  $x = f(y)$ , and let  $U_x \subset \overline{\{x\}}$  be a dense open subset such that  $p(x') \leq p(x) + 2 \text{codim}(x', x)$ . We prove that for  $y' \in U_y \cap f^{-1}(U_x)$ ,  $f_q^* p(y') \leq f_q^* p(y) + 2 \text{codim}(y', y)$ . We may assume that  $p(x) \in \mathbb{Z}$ . Let  $x' = f(y')$ . We have  $f_q^* p(y) = p(x) - q(\text{tr.deg}[k(y) : k(x)])$  and  $f_q^* p(y') = p(x') - q(\text{tr.deg}[k(y') : k(x')])$ . Moreover, by [1, 6.1.2],

$$\delta(y', y) = \text{codim}(y', y) - \text{codim}(x', x).$$

Therefore,

$$\begin{aligned} f_q^* p(y') - f_q^* p(y) &= p(x') - p(x) + q(\text{tr.deg}[k(y) : k(x)]) - q(\text{tr.deg}[k(y') : k(x')]) \\ &\leq 2 \text{codim}(x', x) + 2\delta(y', y) = 2 \text{codim}(y', y). \end{aligned}$$

In other words,  $f^*p$  is an admissible perversity function on  $Y$ .

(3) This is essentially proved in [24, 2.5.2]. □

**Definition 3.1.6** (Pointed schematic neighborhood). Let  $X$  be a higher Artin (resp. Deligne–Mumford) stack. A *pointed smooth (resp. étale) schematic neighborhood* of  $X$  is a triple  $(X_0, u_0, x_0)$  where  $u_0: X_0 \rightarrow X$  is a smooth (resp. an étale) morphism with  $X_0$  in  $\text{Sch}^{\text{qc.sep}}$ , and  $x_0 \in |X_0|$  is a scheme-theoretical point. A morphism  $v: (X_1, u_1, x_1) \rightarrow (X_0, u_0, x_0)$  of pointed smooth (resp. étale) schematic neighborhoods is a smooth (resp. an étale) morphism  $v: X_1 \rightarrow X_0$  such that there is a triangle

$$(3.1) \quad \begin{array}{ccc} X_1 & \xrightarrow{v} & X_0 \\ & \searrow u_1 \quad \swarrow u_0 & \\ & X & \end{array}$$

and  $v(x_1) = x_0$ . We say  $(X_1, u_1, x_1)$  *dominates*  $(X_0, u_0, x_0)$  if there is such a morphism. The category of pointed smooth (resp. étale) schematic neighborhoods of  $X$  is denoted by  $\text{Vo}^{\text{sm}}(X)$  (resp.  $\text{Vo}^{\text{ét}}(X)$ ).

**Lemma 3.1.7.** *Let  $X$  be a higher Artin stack, and let  $v: (X_1, u_1, x_1) \rightarrow (X_0, u_0, x_0)$  be a morphism of pointed smooth schematic neighborhoods of  $X$ . Then the codimension of  $x_1$  in the base change scheme  $X_{1,x_0} = X_1 \times_{X_0} \{x_0\}$  depends only on the source and the target of  $v$ .*

We will denote by  $\delta_{(X_0, u_0, x_0)}^{(X_1, u_1, x_1)}$  this codimension. It is clear that  $\delta_{(X_0, u_0, x_0)}^{(X_2, u_2, x_2)} = \delta_{(X_1, u_1, x_1)}^{(X_2, u_2, x_2)} + \delta_{(X_0, u_0, x_0)}^{(X_1, u_1, x_1)}$  if  $(X_2, u_2, x_2)$  dominates  $(X_1, u_1, x_1)$ . Moreover, if  $v$  is étale,  $\delta_{(X_0, u_0, x_0)}^{(X_1, u_1, x_1)} = 0$ .

*Proof.* Note that  $\text{codim}(x_1, X_{1,x_0}) = \dim_{x_1}(v) - \text{tr.deg}[k(x_1) : k(x_0)]$ . It is clear that  $\dim_{x_1} v = \dim_{x_1} u_1 - \dim_{x_0} u_0$  does not depend on  $v$ . We will show that  $\text{tr.deg}[k(x_1) : k(x_0)]$  does not depend



on  $v$  either. Let  $f: Y \rightarrow X$  be an atlas of  $X$  with  $Y$  a scheme in  $\text{Sch}^{\text{qc.sep}}$ . Let

$$\begin{array}{ccc} Y_1 & \xrightarrow{v'} & Y_0 \\ & \searrow u'_1 \quad \swarrow u'_0 & \\ & Y & \end{array}$$

be the base change of (3.1),  $f_0: Y_0 \rightarrow X_0$  and  $f_1: Y_1 \rightarrow X_1$ . Let  $w_0: Y'_0 \rightarrow Y_0$  be an atlas with  $Y'_0$  a scheme in  $\text{Sch}^{\text{qc.sep}}$ , and let

$$\begin{array}{ccc} Y'_1 & \xrightarrow{v''} & Y'_0 \\ w_1 \downarrow & & \downarrow w_0 \\ Y_1 & \xrightarrow{v'} & Y_0 \end{array}$$

be the base change. Then  $v''$  is a smooth morphism of schemes in  $\text{Sch}^{\text{qc.sep}}$ . Since  $f_0 \circ w_0: Y'_0 \rightarrow X_0$  is smooth and surjective, the base change scheme  $Y'_{0,x_0} = Y'_0 \times_{X_0} \{x_0\}$  is nonempty and smooth over the residue field  $k(x_0)$  of  $x_0$ . Similarly, we have a nonempty scheme  $Y'_{1,x_1}$ , smooth over  $k(x_1)$ . Choose a generic point  $y'_1$  of  $Y'_{1,x_1}$ . Then its image  $y'_0$  in  $Y'_{0,x_0}$  is a generic point. Let  $y$  be the image of  $y'_0$  in  $Y$ . Then  $\text{tr.deg}[k(x_1) : k(x_0)] = \text{tr.deg}[k(y'_1) : k(y)] - \text{tr.deg}[k(y'_0) : k(y)]$ , which does *not* depend on  $v$ .  $\square$

For a higher Artin (resp. Deligne–Mumford) stack  $X$  and a function  $\mathbf{p}: \text{Ob}(\text{Vo}^{\text{sm}}(X)) \rightarrow \mathbb{Z} \cup \{+\infty\}$  (resp.  $\mathbf{p}: \text{Ob}(\text{Vo}^{\text{ét}}(X)) \rightarrow \mathbb{Z} \cup \{+\infty\}$ ), we have, by restriction, the function  $\mathbf{p}_{u_0}: |X_0| \rightarrow \mathbb{Z} \cup \{+\infty\}$  for every smooth (resp. étale) morphism  $u_0: X_0 \rightarrow X$  with  $X_0$  in  $\text{Sch}^{\text{qc.sep}}$ . If  $f: Y \rightarrow X$  is a smooth (resp. an étale) morphism of higher Artin (resp. Deligne–Mumford) stacks, then composition with  $f$  induces a functor  $f: \text{Vo}^{\text{sm}}(Y) \rightarrow \text{Vo}^{\text{sm}}(X)$  (resp.  $f: \text{Vo}^{\text{ét}}(Y) \rightarrow \text{Vo}^{\text{ét}}(X)$ ), and we let  $f^*\mathbf{p} = \mathbf{p} \circ f$ .

*Definition 3.1.8* ((admissible/codimension) perversity evaluations). Let  $X$  be a higher Artin stack. A *smooth evaluation* on  $X$  is a function  $\mathbf{p}: \text{Ob}(\text{Vo}^{\text{sm}}(X)) \rightarrow \mathbb{Z} \cup \{+\infty\}$  such that for  $(X_1, u_1, x_1)$  dominating  $(X_0, u_0, x_0)$ , we have  $\mathbf{p}(X_0, u_0, x_0) \leq \mathbf{p}(X_1, u_1, x_1) \leq \mathbf{p}(X_0, u_0, x_0) + 2\delta_{(X_0, u_0, x_0)}^{(X_1, u_1, x_1)}$ .

A *perversity smooth evaluation* (resp. *admissible perversity smooth evaluation*, *codimension perversity smooth evaluation*) on  $X$  is an evaluation  $\mathbf{p}$  such that for every  $(X_0, u_0, x_0) \in \text{Ob}(\text{Vo}^{\text{sm}}(X))$ ,  $\mathbf{p}_{u_0}$  is a weak perversity function (resp. admissible perversity function, codimension perversity function) on  $X_0$ .

Similarly, we define étale evaluations and (admissible/codimension) perversity étale evaluations on a higher Deligne–Mumford stack  $X$  using  $\text{Vo}^{\text{ét}}(X)$ .

We say that a smooth (resp. étale) evaluation  $\mathbf{p}$  is *locally bounded* if for every smooth (resp. étale) morphism  $u_0: X_0 \rightarrow X$  with  $X_0$  a quasi-compact separated scheme,  $\mathbf{p}_{u_0}$  is bounded.

*Remark 3.1.9.* If  $X$  is a scheme in  $\text{Sch}^{\text{qc.sep}}$ , then the map from the set of étale evaluations on  $X$  to the set of functions  $|X| \rightarrow \mathbb{Z} \cup \{+\infty\}$ , carrying  $\mathbf{p}$  to  $\mathbf{p}_{\text{id}_X}$ , is bijective. Under this bijection, the notions of (weak) perversity, admissible perversity, and codimension perversity coincide. If  $f: Y \rightarrow X$  is a morphism of schemes in  $\text{Sch}^{\text{qc.sep}}$ , then  $f^*$  for étale evaluations coincide with  $f^*_0$  for functions.

*Example 3.1.10.*

- (1) Let  $X$  be a higher Artin (resp. Deligne–Mumford) stack. Any constant smooth (resp. étale) evaluation is an admissible perversity smooth (resp. étale) evaluation.
- (2) Let  $f: Y \rightarrow X$  be a morphism of higher Deligne–Mumford stacks, locally of finite type, let  $\mathbf{p}$  be an étale evaluation on  $X$ , and let  $q: \mathbb{N} \rightarrow \mathbb{Z}$  be a function. We define an étale evaluation  $f_q^*\mathbf{p}$  on  $Y$  as follows. For any object  $(Y_0, v_0, y_0)$  of  $\text{Vo}^{\text{ét}}(Y)$ , there exists a morphism

$(Y_1, v_1, y_1) \rightarrow (Y_0, v_0, y_0)$  in  $\mathrm{Vo}^{\mathrm{ét}}(Y)$  such that there exists a diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{v_1} & Y \\ f_0 \downarrow & & \downarrow f \\ X_0 & \xrightarrow{u_0} & X, \end{array}$$

where  $X_0$  is in  $\mathrm{Sch}^{\mathrm{qc.sep}}$  and  $u_0$  is étale. We put  $f_q^* \mathbf{p}(Y_0, v_0, y_0) = \mathbf{p}(X_0, u_0, f_0(y_1)) - q(\mathrm{tr.deg}[k(y_1) : k(f_0(y_1))])$ . This clearly does not depend on choices. If  $\mathbf{p}$  is a perversity étale evaluation, then  $f_0^* \mathbf{p}$  is a perversity étale evaluation by Lemma 3.1.4.

- (3) Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks with  $X$  being a higher Deligne–Mumford stack, locally of finite type, let  $\mathbf{p}$  be an étale evaluation on  $X$ , and let  $q: \mathbb{Z} \rightarrow \mathbb{Z}$  be an admissible function. We define a smooth evaluation  $f_q^* \mathbf{p}$  on  $Y$  by  $(f_q^* \mathbf{p})(Y_0, v_0, y_0) = (v_0 \circ f)_{q'}^*(y_0)$  for every object  $(Y_0, v_0, y_0)$  of  $\mathrm{Vo}^{\mathrm{sm}}(Y)$ , where  $q': \mathbb{N} \rightarrow \mathbb{Z}$  is the function  $q'(n) = q(n - \dim_{y_0} v_0)$ . If  $\mathbf{p}$  is a perversity étale evaluation, then  $f_0^* \mathbf{p}$  is a perversity smooth evaluation. If  $X$  is locally Noetherian and  $\mathbf{p}$  is a perversity (resp. admissible perversity, resp. codimension perversity) étale evaluation, then  $f_q^* \mathbf{p}$  (resp.  $f_q^* \mathbf{p}$ , resp.  $f_1^* \mathbf{p}$ ) is a perversity (resp. admissible perversity, resp. codimension) smooth evaluation by Lemma 3.1.5.

If  $X$  is an object of  $\mathrm{Sch}^{\mathrm{qc.sep}}$  and  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  is a function, we denote by  $f_q^* p$  the smooth evaluation  $f_q^* \mathbf{p}$  on  $Y$ , where  $\mathbf{p}$  is the perversity étale evaluation corresponding to  $p$ .

**3.2. Perverse  $t$ -structures.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. We say that  $\mathcal{C}$  is *weakly left complete* (resp. *weakly right complete*) if  $\mathcal{C}^{\leq -\infty} = \bigcap_n \mathcal{C}^{\leq -n}$  (resp.  $\mathcal{C}^{\geq \infty} = \bigcap_n \mathcal{C}^{\geq n}$ ) consists of zero objects. The family  $(\mathrm{H}^i)_{i \in \mathbb{Z}}$  is conservative if and only if  $\mathcal{C}$  is weakly left complete and weakly right complete (cf. [5, 1.3.7]). The following generalizes [20, 1.2.1.19].

**Lemma 3.2.1.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. Consider the following conditions*

- (1) *The  $\infty$ -category  $\mathcal{C}$  is left complete (see the definition preceding [20, 1.2.1.18]).*
- (2) *The  $\infty$ -category  $\mathcal{C}$  is weakly left complete.*

*Then (1) implies (2). Moreover, if  $\mathcal{C}$  admits countable products and there exists an integer  $a$  such that countable products of objects of  $\mathcal{C}^{\leq 0}$  belong to  $\mathcal{C}^{\leq a}$ , then (2) implies (1).*

*Proof.* The first assertion is obvious since the image of  $\mathcal{C}^{\leq -\infty}$  under the functor  $\mathcal{C} \rightarrow \hat{\mathcal{C}}$  consists of zero objects.

To show the second assertion, it suffices to replace  $f(n-1)$  by  $f(n-a-1)$  in the proof of [20, 1.2.1.19].  $\square$

Let  $X$  be a scheme  $X$  in  $\mathrm{Sch}^{\mathrm{qc.sep}}$ , let  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  be a function, and let  $\lambda = (\Xi, \Lambda)$  be an object of  $\mathrm{Rind}$ . Following Gabber [9, §2], we define full subcategories  ${}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$  as follows. For  $\mathcal{K}$  in  $\mathcal{D}(X, \lambda)$ ,

- $\mathcal{K}$  belongs to  ${}^p\mathcal{D}^{\leq 0}(X, \lambda)$  if and only if  $i_{\bar{x}}^* j_{\bar{x}}^* \mathcal{K} \in \mathcal{D}^{\leq p(x)}(\bar{x}, \lambda)$  for all  $x \in X$ ;
- $\mathcal{K}$  belongs to  ${}^p\mathcal{D}^{\geq 0}(X, \lambda)$  if and only if  $\mathcal{K} \in \mathcal{D}^{(+)}(X, \lambda)$  and  $i_{\bar{x}}^! j_{\bar{x}}^* \mathcal{K} \in \mathcal{D}^{\geq p(x)}(\bar{x}, \lambda)$  for all  $x \in X$ .

Here  $\bar{x}$  is a geometric point above  $x$ ,  $i_{\bar{x}}: \bar{x} \rightarrow X(\bar{x})$ ,  $j_{\bar{x}}: X(\bar{x}) \rightarrow X$ . We will omit  $j_{\bar{x}}^*$  from the notation when no confusion arises. For an immersion  $i: Z \rightarrow X$  in  $\mathrm{Sch}^{\mathrm{qc.sep}}$ ,  $z \in Z$ , we have an equivalence  $i_{\bar{z}}^! \simeq i_{\bar{z}}^! i^!$  of functors  $\mathcal{D}^{(+)}(X, \lambda) \rightarrow \mathcal{D}^{+}(\bar{z}, \lambda)$ .

Gabber showed in [9] that if  $p$  is a weak perversity function and  $\Xi = *$ , then  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  is a  $t$ -structure on  $\mathcal{D}(X, \lambda)$ . This generalizes easily to the case of general  $\Xi$  as follows. By [20, 1.4.4.11], there exists a  $t$ -structure  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), \mathcal{D}')$  on  $\mathcal{D}(X, \lambda)$ . For  $\mathcal{K} \in {}^p\mathcal{D}^{\leq 0}(X, \lambda)$ ,  $\mathcal{L} \in {}^p\mathcal{D}^{\geq 0}(X, \lambda)$ , we have  $a_* \mathbf{Hom}(\mathcal{K}, \mathcal{L}[1]) \in \mathcal{D}^{\geq 1}(\lambda)$ , so that  $\mathrm{Hom}(\mathcal{K}, \mathcal{L}[1]) = \mathrm{H}^0(\Xi, a_* \mathbf{Hom}(\mathcal{K}, \mathcal{L}[1])) = 0$ , where  $a: X_{\mathrm{ét}} \rightarrow *$  is the morphism of topoi.

Thus  ${}^p\mathcal{D}^{\geq 0}(X, \lambda) \subseteq \mathcal{D}'$ . For every  $\xi \in \Xi$ , the functor  $\mathbf{L}e_{\xi}: \mathcal{D}(X, \Lambda(\xi)) \rightarrow \mathcal{D}(X, \lambda)$  is left  $t$ -exact for the  $t$ -structures  $({}^p\mathcal{D}^{\leq 0}(X, \Lambda(\xi)), {}^p\mathcal{D}^{\geq 0}(X, \Lambda(\xi)))$  and  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), \mathcal{D}')$ . It follows that  $e_{\xi}^*$  is right  $t$ -exact for the same  $t$ -structures. Therefore  $\mathcal{D}' \subseteq {}^p\mathcal{D}^{\geq 0}(X, \lambda)$ .

Thus  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  is a  $t$ -structure on  $\mathcal{D}(X, \lambda)$ . By definition, this  $t$ -structure is accessible, and, if  $p$  takes values in  $\mathbb{Z}$ , weakly left complete. By [9, 3.1], this  $t$ -structure is weakly right complete, thus right complete. By Lemma 3.2.1, the above  $t$ -structure is left complete if  $p$  is locally bounded and every quasi-compact closed open subscheme of  $X$  is  $\lambda$ -cohomologically finite. We say that a scheme  $Y$  is  $\lambda$ -cohomologically finite if there exists an integer  $n$  such that, for every  $\xi \in \Xi$ , the  $\lambda(\xi)$ -cohomological dimension of the étale topos of  $Y$  is at most  $n$ .

By definition, for any morphism  $f: Y \rightarrow X$  of schemes in  $\text{Sch}^{\text{qc.sep}}$ ,  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  carries  ${}^p\mathcal{D}^{\leq 0}(X, \lambda)$  to  $f_0^* {}^p\mathcal{D}^{\leq 0}(Y, \lambda)$ . Moreover, if  $f$  is étale, then  $f^*$  carries  ${}^p\mathcal{D}^{\geq 0}(X, \lambda)$  to  $f_0^* {}^p\mathcal{D}^{\geq 0}(Y, \lambda)$ . We will prove an analogue of this for smooth morphisms in Proposition 3.2.5.

**Lemma 3.2.2.** *Let  $f: Y \rightarrow X$  be a smooth morphism in  $\text{Sch}_{\mathbb{L}}^{\text{qc.sep}}$ , let  $\lambda$  be an object of  $\text{Rind}_{\mathbb{L}\text{-tor}}$ , and let  $y$  be a point of  $Y$ ,  $x = f(y)$ . Then there is an equivalence  $i_{\bar{y}}^! \circ f^! \simeq g^* \circ i_{\bar{x}}^! \langle d \rangle$  of functors  $\mathcal{D}^{(+)}(X, \lambda) \rightarrow \mathcal{D}^{+}(\bar{y}, \lambda)$ , where  $g: \bar{y} \rightarrow \bar{x}$ ,  $d = \text{tr.deg}[k(y) : k(x)]$ .*

*Proof.* Consider the diagram with Cartesian squares

$$\begin{array}{ccccccc} \bar{y} & \xrightarrow{i_{\bar{y}}} & V & \xrightarrow{j} & Y_{\bar{x}} & \xrightarrow{i'_{\bar{x}}} & Y_{\{x\}} & \xrightarrow{i'} & Y \\ & \searrow g & & & \downarrow f_{\bar{x}} & & \downarrow f_{\{x\}} & & \downarrow f \\ & & & & \bar{x} & \xrightarrow{i_{\bar{x}}} & \{x\} & \xrightarrow{i} & X \end{array}$$

where  $V$  is a regular integral subscheme of  $Y_{\bar{x}}$  such that the image of  $\bar{y}$  in  $V$  is a generic point. We have a sequence of equivalences of functors

$$\begin{aligned} i_{\bar{y}}^! \circ f^! &\simeq i_{\bar{y}}^* \circ j^! \circ i'_{\bar{x}}^* \circ i'^! \circ f^! \simeq i_{\bar{y}}^* \circ j^! \circ i'_{\bar{x}}^* \circ f_{\{x\}}^! \circ i^! \\ &\simeq i_{\bar{y}}^* \circ j^! \circ f_{\bar{x}}^! \circ i_{\bar{x}}^* \circ i^! && \text{by Poincaré duality} \\ &\simeq i_{\bar{y}}^* \circ (f_{\bar{x}} \circ j)^! \circ i_{\bar{x}}^! \\ &\simeq i_{\bar{y}}^* \circ (f_{\bar{x}} \circ j)^* \circ i_{\bar{x}}^! \langle d \rangle && \text{by Poincaré duality} \\ &\simeq g^* \circ i_{\bar{x}}^! \langle d \rangle. \end{aligned}$$

□

**Lemma 3.2.3.** *Let  $\lambda$  be an object of  $\text{Rind}_{\mathbb{L}\text{-tor}}$ , let  $f: Y \rightarrow X$  be a smooth morphism of schemes in  $\text{Sch}_{\mathbb{L}}^{\text{qc.sep}}$ , and let  $p: |X| \rightarrow \mathbb{Z} \cup \{+\infty\}$  be a function. Then  $f^!$  carries  ${}^p\mathcal{D}^{\geq 0}(X, \lambda)$  to  $f_2^* {}^p\mathcal{D}^{\geq 0}(Y, \lambda)$ . Moreover, if  $p$  is a weak perversity function on  $X$  and  $q$  is a weak perversity function on  $Y$  satisfying  $f_0^* p \leq q \leq f_2^* p + 2 \dim f$ , then  $f^!: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is  $t$ -exact with respect to the  $t$ -structures associated to  $p$  and  $q$ .*

*Proof.* The first assertion follows from the above lemma. The second assertion follows from the first assertion and Poincaré duality  $f^! \simeq f^* \langle \dim f \rangle$ . □

Let  $X$  be an  $\mathbb{L}$ -coprime higher Artin (resp. a higher Deligne–Mumford) stack equipped with a perversity evaluation  $\mathbf{p}$ , and let  $\lambda$  be an object of  $\text{Rind}_{\mathbb{L}\text{-tor}}$  (resp.  $\text{Rind}$ ). For an atlas (resp. étale atlas)  $u: X_0 \rightarrow X$  with  $X_0$  a scheme in  $\text{Sch}^{\text{qc.sep}}$ , we denote by  ${}^p\mathcal{D}_u^{\leq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$  (resp.  ${}^p\mathcal{D}_u^{\geq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$ ) the full subcategory spanned by complexes  $\mathcal{K}$  such that  $u^* \mathcal{K}$  is in  ${}^p\mathcal{D}^{\leq 0}(X_0, \lambda)$  (resp.  ${}^p\mathcal{D}^{\geq 0}(X_0, \lambda)$ ).

**Lemma 3.2.4.** *The pair of subcategories  $({}^p\mathcal{D}_u^{\leq 0}(X, \lambda), {}^p\mathcal{D}_u^{\geq 0}(X, \lambda))$  do not depend on the choice of  $u$ .*

In what follows, we will write  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  for  $({}^p\mathcal{D}_u^{\leq 0}(X, \lambda), {}^p\mathcal{D}_u^{\geq 0}(X, \lambda))$ .

**Proposition 3.2.5.** *Let  $X$  be an  $\mathbb{L}$ -coprime higher Artin (resp. a higher Deligne–Mumford) stack equipped with a perversity smooth (resp. étale) evaluation  $\mathfrak{p}$ , and let  $\lambda$  be an object of  $\mathcal{R}\mathrm{ind}_{\mathbb{L}\text{-tor}}$  (resp.  $\mathcal{R}\mathrm{ind}$ ). Then*

- (1) *The pair of subcategories  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  determine a right complete accessible  $t$ -structure on  $\mathcal{D}(X, \lambda)$ , which is weakly left complete if  $\mathfrak{p}$  takes values in  $\mathbb{Z}$ . This  $t$ -structure is left complete if  $\mathfrak{p}$  is locally bounded and if for every smooth (resp. étale) morphism  $X_0 \rightarrow X$  with  $X_0$  a quasi-compact separated scheme,  $X_0$  is  $\lambda$ -cohomologically finite.*
- (2) *If  $f: Y \rightarrow X$  is a smooth (resp. étale) morphism, then  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is  $t$ -exact with respect to the  $t$ -structures associated to  $\mathfrak{p}$  and  $f^*\mathfrak{p}$ .*

*Proof of Lemma 3.2.4 and Proposition 3.2.5.* There exists  $k \geq 2$  such that  $X$  and  $Y$  are in  $\mathrm{Chp}^{k\text{-Ar}}$  (resp.  $\mathrm{Chp}^{k\text{-DM}}$ ). We proceed by induction on  $k$ . The case  $k = -2$  follows from Gabber’s theorem and Lemma 3.2.3. The induction step follows from the proofs of [18, 4.3.7, 4.3.8].  $\square$

*Remark 3.2.6.* We call the  $t$ -structure in Proposition 3.2.5 the *perverse  $t$ -structure* with respect to  $\mathfrak{p}$  and denote by  ${}^p\tau^{\leq 0}$  and  ${}^p\tau^{\geq 0}$  the corresponding truncation functors respectively. The for every (étale) atlas  $u: X_0 \rightarrow X$  with  $X_0$  a scheme in  $\mathrm{Sch}^{\mathrm{qc}, \mathrm{sep}}$ ,  $u^* \circ {}^p\tau^{\leq 0} \simeq {}^p\tau^{\leq 0} \circ u$  and  $u^* \circ {}^p\tau^{\geq 0} \simeq {}^p\tau^{\geq 0} \circ u$ .

If  $\mathfrak{p} = 0$ , then we recover the usual  $t$ -structure. If  $X$  is a higher Deligne–Mumford stack and  $\mathfrak{p}$  is a perversity smooth evaluation, then the  $t$ -structure associated to  $\mathfrak{p}$  coincides with the  $t$ -structure associated to  $\mathfrak{p} \mid \mathrm{Vo}^{\mathrm{ét}}(X)$ . If  $X$  is in  $\mathrm{Sch}^{\mathrm{qc}, \mathrm{sep}}$ , then the  $t$ -structure associated to  $\mathfrak{p}$  coincides with the  $t$ -structure defined by Gabber associated to the function  $\mathfrak{p}_{\mathrm{id}_X}$ .

By definition, the perverse  $t$ -structure can be described as follows.

**Proposition 3.2.7.** *Let  $X$  be an  $\mathbb{L}$ -coprime higher Artin stack (resp. a higher Deligne–Mumford stack) equipped with a perversity smooth (resp. étale) evaluation  $\mathfrak{p}$ , and let  $\lambda$  be an object of  $\mathcal{R}\mathrm{ind}_{\mathbb{L}\text{-tor}}$  (resp.  $\mathcal{R}\mathrm{ind}$ ). Let  $\mathcal{K}$  be a complex in  $\mathcal{D}(X, \lambda)$ .*

- (1)  *$\mathcal{K}$  belongs to  ${}^p\mathcal{D}^{\leq n}(X, \lambda)$  if and only if for every pointed smooth (resp. étale) schematic neighborhood  $(X_0, u_0, x_0)$  of  $X$ ,  $i_{x_0}^* u_0^* \mathcal{K} \in \mathcal{D}^{\leq \mathfrak{p}(X_0, u_0, x_0) + n}(\overline{x_0}, \lambda)$ .*
- (2)  *$\mathcal{K}$  belongs to  ${}^p\mathcal{D}^{\geq n}(X, \lambda)$  if and only if  $\mathcal{K} \in \mathcal{D}^{(+)}(X, \lambda)$  and for every pointed smooth (resp. étale) schematic neighborhood  $(X_0, u_0, x_0)$  of  $X$ ,  $i_{x_0}^! u_0^* \mathcal{K} \in \mathcal{D}^{\geq \mathfrak{p}(X_0, u_0, x_0) + n}(\overline{x_0}, \lambda)$ .*

At the end of the section, we study the restriction of perverse  $t$ -structures constructed above to various subcategory of constructible complexes. We fix an  $\mathbb{L}$ -coprime base scheme  $S$  that is a disjoint union of excellent schemes, endowed with a global dimension function.

**Proposition 3.2.8.** *Let  $\lambda = (\Xi, \Lambda)$  be an object of  $\mathcal{R}\mathrm{ind}_{\mathbb{L}\text{-dual}}$ . Let  $f: X \rightarrow S$  be an object of  $\mathrm{Chp}_{\mathrm{ift}/S}^{\mathrm{Ar}}$  equipped with an admissible perversity smooth evaluation  $\mathfrak{p}$ . Then the truncation functors  ${}^p\tau^{\leq 0}, {}^p\tau^{\geq 0}$  preserve  $\mathcal{D}_{\mathrm{cons}}^{(b)}(X, \lambda)$ . Moreover, if  $\mathfrak{p}$  is a locally bounded, then  ${}^p\tau^{\leq 0}, {}^p\tau^{\geq 0}$  preserve  $\mathcal{D}_{\mathrm{cons}}^?(X, \lambda)$  for  $? = (+), (-)$  or empty.*

*Proof.* We reduce easily to the case of a scheme. In this case, the result is essentially [9, 8.2].  $\square$

**3.3. Adic perverse  $t$ -structures.** For the adic formalism, we define

$${}^p\mathcal{D}^{\leq n}(X, \lambda) = {}^p\mathcal{D}^{\leq n}(X, \lambda) \cap \mathcal{D}(X, \lambda), \quad {}^p\mathcal{D}^{\geq n}(X, \lambda) = {}^p\mathcal{D}^{\leq n-1}(X, \lambda)^\perp \subseteq \mathcal{D}(X, \lambda).$$

Then the pair  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  defines a  $t$ -structure, called the *adic perverse  $t$ -structure* with respect to  $\mathfrak{p}$ , on  $\mathcal{D}(X, \lambda)$ . Denote  ${}^p\tau^{\leq 0}$  and  ${}^p\tau^{\geq 0}$  the corresponding truncation functors respectively. We first have the following.

**Lemma 3.3.1.** *Let  $X$  be an  $\mathbb{L}$ -coprime higher Artin stack (resp. a higher Deligne–Mumford stack) equipped with a perversity smooth (resp. étale) evaluation  $\mathfrak{p}$ , and  $\lambda$  be an object of  $\mathcal{R}\mathrm{ind}_{\mathbb{L}\text{-tor}}$  (resp.*

Rind). Let  $\mathcal{K}$  be a complex in  $\underline{\mathcal{D}}(X, \lambda)$ . Let  $u: X_0 \rightarrow X$  be an atlas (resp. étale atlas) with  $X_0$  a scheme in  $\text{Sch}^{\text{qc.sep}}$ . Then  $\mathcal{K}$  belongs to  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$  (resp.  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq n}(X, \lambda)$ ) if and only if  $\underline{u}^*\mathcal{K}$  belongs to  ${}^{\mathbf{p}_u}\underline{\mathcal{D}}^{\leq n}(X_0, \lambda)$  (resp.  ${}^{\mathbf{p}_u}\underline{\mathcal{D}}^{\geq n}(X_0, \lambda)$ ).

*Proof.* We only need to show that  $\underline{u}^*$  is  $t$ -exact. By definition, we obviously have  $\underline{u}^*{}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda) \subseteq {}^{\mathbf{p}_u}\underline{\mathcal{D}}^{\leq n}(X_0, \lambda)$ . For the other direction, assume  $\mathcal{K} \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq n}(X, \lambda)$ , that is,  $\text{Hom}(\mathcal{L}, \mathcal{K}) = 0$  for all  $\mathcal{L} \in \underline{\mathcal{D}}(X, \lambda) \cap {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$ . By Poincaré duality, we only need to show that for  $\mathcal{L}' \in \underline{\mathcal{D}}(X_0, \lambda) \cap {}^{\mathbf{p}_u}\underline{\mathcal{D}}^{\leq n-2\dim u}(X_0, \lambda)$ ,  $\text{Hom}(\mathcal{L}', u^!\mathcal{K}) = 0$ , or equivalently,  $\text{Hom}(u_!\mathcal{L}', \mathcal{K}) = 0$ . This follows from the fact that  $u_!$  preserves adic objects and  $u_!\mathcal{L}' \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$ .  $\square$

We have the following description in terms of the cohomology on stalks, that is similar to the Proposition 3.2.7.

**Proposition 3.3.2.** *Let  $X$  be an  $\mathbf{L}$ -coprime higher Artin stack (resp. a higher Deligne–Mumford stack) equipped with a perversity smooth (resp. étale) evaluation  $\mathbf{p}$ , and  $\lambda$  be an object of  $\text{Rind}_{\mathbf{L}\text{-tor}}$  (resp.  $\text{Rind}$ ). Let  $\mathcal{K}$  be a complex in  $\underline{\mathcal{D}}(X, \lambda)$ .*

- (1)  $\mathcal{K}$  belongs to  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$  if and only if for every pointed smooth (resp. étale) schematic neighborhood  $(X_0, u_0, x_0)$  of  $X$ ,  $i_{\overline{x}_0}^*u_0^*\mathcal{K} \in \underline{\mathcal{D}}^{\leq \mathbf{p}(X_0, u_0, x_0)+n}(\overline{x}_0, \lambda)$ .
- (2) Assume  $\mathbf{p}$  is locally bounded. Then  $\mathcal{K}$  belongs to  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq n}(X, \lambda)$  if and only if  $\mathcal{K} \in \underline{\mathcal{D}}^{(+)}(X, \lambda)$  and for every pointed smooth (resp. étale) schematic neighborhood  $(X_0, u_0, x_0)$  of  $X$ ,  $i_{\overline{x}_0}^!u_0^*\mathcal{K} \in \underline{\mathcal{D}}^{\geq \mathbf{p}(X_0, u_0, x_0)+n}(\overline{x}_0, \lambda)$ .

*Proof.* For (1), by definition,  $\mathcal{K}$  belongs to  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$  if and only if  $\mathcal{K} \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$ , viewed as object of  $\mathcal{D}(X, \lambda)$ . By Proposition 3.2.7 (1), it is equivalent to say that for  $(Y, u, y)$  of  $X$ ,  $i_y^*u^*\mathcal{K} \in \underline{\mathcal{D}}^{\leq \mathbf{p}(Y, u, y)+n}(\overline{y}, \lambda)$ . We only need to notice that  $i_y^*u^*\mathcal{K}$  is adic.

For (2), by Lemma 3.3.1, we may assume  $X \in \text{Sch}^{\text{qc.sep}}$  is quasi-compact and  $\mathbf{p} = p$  is a bounded weak perversity function. Then  $\mathcal{K} \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq n}(X, \lambda)$  is equivalent to that for every  $\mathcal{L} \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda)$ ,  $\mathbf{Hom}(\mathcal{L}, \mathcal{K}) \in \mathcal{D}^{>0}(X, \lambda)$ , which is then equivalent to that  $\mathbf{Hom}(\mathcal{L}, \mathcal{K}) \in \mathcal{D}^+(X, \lambda)$  and  $i_{\overline{x}}^!\mathbf{Hom}(\mathcal{L}, \mathcal{K}) \simeq \mathbf{Hom}(i_{\overline{x}}^*\mathcal{L}, i_{\overline{x}}^!\mathcal{K}) \simeq \mathbf{Hom}(i_{\overline{x}}^*\mathcal{L}, i_{\overline{x}}^!\mathcal{K}) \in \mathcal{D}^{>0}(\overline{x}, \lambda)$  for every geometric point  $\overline{x}$  of  $X$ . Assume  $\alpha < p < \beta$ . Then  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq n}(X, \lambda) \supseteq \underline{\mathcal{D}}^{\leq \alpha+n}(X, \lambda)$ . Therefore,  $\mathcal{K} \in {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq n}(X, \lambda)$  implies  $\mathcal{K} \in \underline{\mathcal{D}}^{\geq \alpha+n}(X, \lambda)$  and  $i_{\overline{x}}^!\mathcal{K} \in \underline{\mathcal{D}}^{\geq p(\overline{x})+n}(\overline{x}, \lambda)$  for every geometric point  $\overline{x}$  of  $X$ . Conversely, assume  $\mathcal{K} \in \underline{\mathcal{D}}^+(X, \lambda)$ , say in  $\underline{\mathcal{D}}^{\geq \gamma}(X, \lambda)$ , and  $i_{\overline{x}}^!\mathcal{K} \in \underline{\mathcal{D}}^{\geq p(\overline{x})+n}(\overline{x}, \lambda)$  for every geometric point  $\overline{x}$  of  $X$ . We only need to show that  $\mathbf{Hom}(\mathcal{L}, \mathcal{K}) \in \mathcal{D}^+(X, \lambda)$ . In fact, we have  $\mathbf{Hom}(\mathcal{L}, \mathcal{K}) \in \mathcal{D}^{\geq \gamma-\beta-n}(X, \lambda)$ .  $\square$

*Remark 3.3.3.* Let  $\mathbf{p}, \mathbf{q}$  be two perversity smooth (resp. étale) evaluations on an  $\mathbf{L}$ -coprime higher Artin stack (resp. a higher Deligne–Mumford stack)  $X$ . Let  $\lambda$  be an object of  $\text{Rind}_{\mathbf{L}\text{-tor}}$  (resp.  $\text{Rind}$ ).

- (1) Since  ${}^{\mathbf{p}}\tau^{\geq 0}$  (resp.  ${}^{\mathbf{p}}\tau^{\leq 0}$ ) preserves  $\mathcal{D}^{(+)}(X, \lambda)$  (resp.  $\underline{\mathcal{D}}^{(+)}(X, \lambda)$ ), so does  ${}^{\mathbf{p}}\tau^{\leq 0}$  (resp.  ${}^{\mathbf{p}}\tau^{\geq 0}$ ). Therefore, the intersection of  $({}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda), {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda))$  (resp.  $({}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda), {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda))$ ) with  $\mathcal{D}^{(+)}(X, \lambda)$  (resp.  $\underline{\mathcal{D}}^{(+)}(X, \lambda)$ ) induces a  $t$ -structure on the latter  $\infty$ -category.
- (2) If  $\mathbf{p} \leq \mathbf{q}$ , then
  - (a)  ${}^{\mathbf{p}}\tau^{\leq 0}$  (resp.  ${}^{\mathbf{p}}\tau^{\geq 0}$ ) preserves  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$ ) since  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda) \subseteq {}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda) \subseteq {}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$ );
  - (b)  ${}^{\mathbf{q}}\tau^{\geq 0}$  (resp.  ${}^{\mathbf{q}}\tau^{\leq 0}$ ) preserves  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$ ) since  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda) \subseteq {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda) \subseteq {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$ );
  - (c)  ${}^{\mathbf{p}}\tau^{\geq 0}$  (resp.  ${}^{\mathbf{p}}\tau^{\leq 0}$ ) is equivalent to the identity function when restricted to  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$ ) since  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda) \subseteq {}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda) \subseteq {}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$ );
  - (d)  ${}^{\mathbf{q}}\tau^{\leq 0}$  (resp.  ${}^{\mathbf{q}}\tau^{\geq 0}$ ) is equivalent to the identity function when restricted to  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$ ) since  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda) \subseteq {}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{p}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda) \subseteq {}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$ );
  - (e)  ${}^{\mathbf{p}}\tau^{\leq 0}$  (resp.  ${}^{\mathbf{p}}\tau^{\geq 0}$ ) is equivalent to the null function when restricted to  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\geq 0}(X, \lambda)$  (resp.  ${}^{\mathbf{q}}\underline{\mathcal{D}}^{\leq 0}(X, \lambda)$ );



- (f)  ${}^q\tau^{>0}$  (resp.  ${}^q\tau^{>0}$ ) is equivalent to the null function when restricted to  ${}^p\mathcal{D}^{\leq 0}(X, \lambda)$  (resp.  ${}^p\mathcal{D}^{\leq 0}(X, \lambda)$ );
- (3) By 2 (a), if  $\mathfrak{p}$  is locally bounded, then the intersection of  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$  (resp.  $({}^p\mathcal{D}^{\leq 0}(X, \lambda), {}^p\mathcal{D}^{\geq 0}(X, \lambda))$ ) with  $\mathcal{D}^{(-)}(X, \lambda)$  or  $\mathcal{D}^{(b)}(X, \lambda)$  (resp.  $\mathcal{D}^{(-)}(X, \lambda)$  or  $\mathcal{D}^{(b)}(X, \lambda)$ ) induces a  $t$ -structure on the latter  $\infty$ -category.
- (4) By 2 (e) and (f), if  $X$  is quasi-compact and  $\mathfrak{p}$  is bounded, then there exist constant integers  $\alpha < \beta$  such that  ${}^p\mathcal{H}^0 = {}^p\mathcal{H}^0 \circ \tau^{[\alpha, \beta]}$  (resp.  ${}^p\mathcal{H}^0 = {}^p\mathcal{H}^0 \circ \tau^{[\alpha, \beta]}$ ).

**3.4. Constructible adic perverse  $t$ -structures.** We fix an  $L$ -coprime base scheme  $S$  that is a disjoint union of schemes that are excellent, quasi-compact, finite dimensional, and admit a global dimension function for which we fix one. We fix also an object  $(\Lambda, \mathfrak{m})$  of  $\mathcal{P}\text{Ring}_{L\text{-tor}}$  such that  $\Lambda/\mathfrak{m}^{n+1}$  is an  $(L\text{-torsion})$  Gorenstein ring of dimension 0 for every  $n \in \mathbb{N}$  and  $S$  is locally  $(\Lambda/\mathfrak{m})$ -bounded.

**Proposition 3.4.1.** *For an object  $f: X \rightarrow S$  of  $\text{Chp}_{\text{ift}/S}^{\text{Ar}}$  equipped with an admissible perversity evaluation  $\mathfrak{p}$ , the truncation functors  ${}^p\tau^{\leq 0}$ ,  ${}^p\tau^{\geq 0}$  preserve  $\mathcal{D}_{\text{cons}}^?(X, \Lambda_{\bullet})$  for  $? = (+), (-), (b)$  or empty.*

*Proof.* By Lemma 3.3.1, we may assume that  $X$  is a quasi-compact, separated (and excellent, finite dimensional) scheme that is  $(\Lambda/\mathfrak{m})$ -bounded, and  $\mathfrak{p} = p$  is an admissible perversity function on  $X$ . In particular,  $p$  is bounded. We prove by Noetherian induction. We may further assume  $X$  is irreducible. For a complex  $\mathcal{K} \in \mathcal{D}_{\text{cons}}(X, \Lambda_{\bullet})$ , we may assume  $\mathcal{K} \in \mathcal{D}_{\text{cons}}^b(X, \Lambda_{\bullet}) \subseteq \mathcal{D}^b(X, \Lambda_{\bullet})$  by Remark 3.3.3 (3). Choose a dense open subset  $U$  of  $X$  such that

- $U$  is essentially smooth;
- $p(x) \leq p(\eta) + \text{codim}_X(x)$  for  $x \in |U|$ , where  $\eta$  is the unique generic point of  $X$ ;
- $p(x) \geq p(\eta)$  for  $x \in |U|$ ;
- The complex  $\mathcal{K}_U := \mathcal{K}|_U$ , viewed as an element of  $\mathcal{D}^b(U, \Lambda_{\bullet})$ , has smooth almost adic cohomology sheaves.

Then the perverse truncation for  $\mathcal{K}_U$  is simply the usual truncation (up to a shift by  $p(\eta)$ ), which preserves constructibility by Corollary 2.4.4.  $\square$

Our definition of the constructible adic perverse  $t$ -structure coincides with Laszlo–Olsson [15] under their restrictions, where in particular  $X$  is a locally Noetherian (1-)Artin stack over a field  $k$  (that is,  $S = \text{Spec } k$ ) with  $\text{cd}_{\ell}(k) < \infty$ , and  $\mathfrak{p}$  is the middle perversity smooth evaluation, that is, the unique perverse smooth evaluation such that for every atlas  $u: X_0 \rightarrow X$  with  $X_0$  a scheme in  $\text{Sch}^{\text{qc.sep}}$ ,  $\mathfrak{p}_u = (f \circ u)_! p_0$ , where  $f: X \rightarrow S$  is the structure morphism and  $p_0$  is the zero perverse function on  $S = \text{Spec } k$ .

#### 4. DESCENT PROPERTIES

In §4.1, we prove that our construction of derived  $\infty$ -categories, as well as those of adic complexes, of (higher) Artin stacks satisfies not only the smooth descent, but also the smooth *hyperdescent*.

**4.1. Hyperdescent.** The étale  $\infty$ -topos of an affine scheme is not hypercomplete in general. By contrast, the stable  $\infty$ -categories we constructed satisfy smooth hyperdescent. We begin with a general definition.

*Definition 4.1.1 ( $F$ -descent).* Let  $\mathcal{C}, \mathcal{D}$  be  $\infty$ -categories, let  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$  be a functor, and let  $X_{\bullet}^+: N(\Delta_+)^{\text{op}} \rightarrow \mathcal{C}$  be an augmented simplicial object of  $\mathcal{C}$ .

- (1) We say that  $X_{\bullet}^+$  is an *augmentation of  $F$ -descent* if  $F \circ (X_{\bullet}^+)^{\text{op}}$  is a limit diagram in  $\mathcal{D}$ .
- (2) Assume  $\mathcal{C}$  admits pullbacks. We say that  $X_{\bullet}^+$  is a *hypercovering for universal  $F$ -descent* if  $X_q^+ \rightarrow (\text{cosk}_{q-1}(X_{\bullet}^+/X_{-1}^+))_q$  is a morphism of universal  $F$ -descent for all  $q \geq 0$ .

By definition, a morphism of  $\mathcal{C}$  is of  $F$ -descent [18, 3.1.1] if and only if its Čech nerve is an augmentation of  $F$ -descent. We now give several criteria for (2)  $\Rightarrow$  (1).

**Proposition 4.1.2.** *Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks, let  $\mathcal{D}$  be an  $n$ -category admitting finite limits for an integer  $n \geq 0$ , and let  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  be a functor. Then every hypercovering  $X_\bullet^+$  for universal  $F$ -descent is an augmentation of  $F$ -descent.*

To prove Proposition 4.1.2, we need a few lemmas.

**Lemma 4.1.3.** *Let  $\mathcal{C}, \mathcal{D}$  be  $\infty$ -categories such that  $\mathcal{C}$  admits finite limits, let  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  be a functor, let  $e$  be a final object of  $\mathcal{C}$ , and let  $f_\bullet: U_\bullet \rightarrow V_\bullet$  be a morphism of simplicial objects of  $\mathcal{C}$  such that  $V_\bullet \rightarrow e$  is an augmentation of  $F$ -descent and  $f_q$  is a morphism of  $F$ -descent for all  $q$ . Assume that there exists an integer  $n \geq 0$  such that  $U_\bullet$  is  $n$ -coskeletal,  $V_\bullet$  is  $(n-1)$ -coskeletal, and  $f_q$  is an equivalence for  $q < n$ . Then  $U_\bullet \rightarrow e$  is an augmentation of  $F$ -descent.*

*Proof.* We may assume  $F(e)$  is an initial object of  $\mathcal{D}$ . Let  $W_+: N(\Delta_+ \times \Delta)^{op} \rightarrow \mathcal{D}$  be a Čech nerve of  $f_\bullet$ ,  $W = W_+ | N(\Delta \times \Delta)^{op}$ . For every  $q \geq 0$ ,  $W_+ | N(\Delta_+ \times \{[q]\})^{op}$  is a Čech nerve of  $f_q$ , which is a morphism of  $F$ -descent by assumption. It follows that  $F \circ W_+^{op} | N(\Delta_+ \times \{[q]\})$  is a limit diagram. We may thus identify the limit of  $F \circ W^{op}$  with the limit  $F \circ W_+^{op} | N(\{[-1]\} \times \Delta_s)$ . Since  $W_+ | N(\{[-1]\} \times \Delta)^{op}$  can be identified with  $V_\bullet$ , the limit of  $F \circ W^{op}$  can be identified with  $F(e)$ . Let  $D_\bullet = W \circ \delta$ , where  $\delta: N(\Delta)^{op} \rightarrow N(\Delta \times \Delta)^{op}$  is the diagonal map. Since  $N(\Delta)^{op}$  is sifted [19, 5.5.8.4], the limit of  $F \circ D_\bullet^{op}$  can be identified with  $F(e)$ . The proof of [19, 6.5.3.9] exhibits  $U_\bullet | N(\Delta_s)^{op}$  as a retract of  $D_\bullet | N(\Delta_s)^{op}$ . It follows that the limit of  $F \circ U_\bullet^{op}$  is a retract of  $F(e)$ , and hence is  $F(e)$ .  $\square$

**Lemma 4.1.4.** *Let  $\mathcal{C}, \mathcal{D}$  be  $\infty$ -categories such that  $\mathcal{C}$  admits pullbacks, let  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  be a functor, and let  $X_\bullet^+$  be an  $n$ -coskeletal hypercovering for universal  $F$ -descent for an integer  $n \geq -1$ . Then  $X_\bullet^+$  is an augmentation of  $F$ -descent.*

*Proof.* Since morphisms of universal  $F$ -descent are stable under pullback and composition,  $\text{cosk}_m(X_\bullet^+/X_{-1}^+) \rightarrow \text{cosk}_{m-1}(X_\bullet^+/X_{-1}^+)$  satisfies the assumptions of Lemma 4.1.3. It follows by induction that  $\text{cosk}_n(X_\bullet^+/X_{-1}^+)$  is an augmentation of  $F$ -descent.  $\square$

**Lemma 4.1.5.** *Let  $n \geq -1$  be an integer, let  $\mathcal{D}$  be an  $n$ -category admitting finite colimits, and let  $f_\bullet: Y_\bullet \rightarrow X_\bullet$  be a morphism of semisimplicial (resp. simplicial) objects of  $\mathcal{D}$  such that  $Y_q \rightarrow X_q$  is an equivalence for  $q \leq n$ . Then the induced morphism between geometric realizations  $|f_\bullet|: |Y_\bullet| \rightarrow |X_\bullet|$  is an equivalence in  $\mathcal{D}$ .*

The existence of the geometric realizations is guaranteed by [20, 1.3.3.10].

*Proof.* The semisimplicial case follows from the simplicial case by taking left Kan extensions. The simplicial case follows from the proof of [20, 1.3.3.10].  $\square$

*Proof of Proposition 4.1.2.* It suffices to apply the dual version of Lemma 4.1.5 to  $h: X_\bullet^+ \rightarrow \text{cosk}_n(X_\bullet^+/X_{-1}^+)$  and Lemma 4.1.4.  $\square$

The following can be used to deduce Gabber's hyper base change theorem [23, Théorème 2.2.5] (see [27, Remark 2.3]).

**Proposition 4.1.6.** *Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks, let  $\mathcal{D}$  be a stable  $\infty$ -category endowed with a weakly right complete  $t$ -structure that either admits countable limits or is right complete, let  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  be a functor, and let  $X_\bullet^+: N(\Delta_+)^{op} \rightarrow \mathcal{C}$  be a hypercovering for universal  $F$ -descent such that  $F \circ (X_\bullet^+)^{op}$  factorizes through  $\mathcal{D}^{\geq 0}$ . Then  $X_\bullet^+$  is an augmentation of  $F$ -descent.*

*Proof.* Let  $n \geq 0$ . By Lemma 4.1.4,  $Y_\bullet^+ = \text{cosk}_n(X_\bullet^+/X_{-1}^+)$  is an augmentation of  $F$ -descent, so that it suffices to show that the morphism  $h_\bullet: X_\bullet^+ \rightarrow Y_\bullet^+$  induces an isomorphism  $c: K =$



$\varprojlim_{p \in \Delta} F(X_p) \rightarrow \varprojlim_{p \in \Delta} F(Y_p) = L$  is an isomorphism. By [20, 1.2.4.4, 1.2.4.5], we have a morphism of converging spectral sequences

$$\begin{array}{ccc} E_1^{pq} = H^q F(X_p) & \Longrightarrow & H^{p+q} K \\ c_1^{pq} \downarrow & & \downarrow H^{p+q} c \\ E_1'^{pq} = H^q F(Y_p) & \Longrightarrow & H^{p+q} L, \end{array}$$

concentrated in the first quadrant. For  $p \leq n$ , since  $h_p$  is an equivalence,  $c_1^{pq}$  is an isomorphism for all  $q$ . It follows that  $c_r^{pq}$  is an isomorphism for  $p + q \leq n - 1$ , and  $\tau^{\leq n-1} c$  is an equivalence. Since  $n$  is arbitrary and  $\mathcal{D}$  is weakly right complete,  $c$  is an equivalence.  $\square$

We denote by  $\mathcal{P}_{\text{st},t}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}}$ ) the  $\infty$ -category defined as follows.

- Objects of  $\mathcal{P}_{\text{st},t}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}}$ ) are presentable stable  $\infty$ -categories  $\mathcal{C}$  equipped with a  $t$ -structure.
- Morphisms of  $\mathcal{P}_{\text{st},t}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}}$ ) are  $t$ -exact functors admitting right (resp. left) adjoints.

The  $\infty$ -categories  $\mathcal{P}_{\text{st},t}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}}$ ) admit small limits, and those limits are preserved by the forgetful functor  $\mathcal{P}_{\text{st},t}^{\text{L}} \rightarrow \mathcal{P}_{\text{st}}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}} \rightarrow \mathcal{P}_{\text{st}}^{\text{R}}$ ). For a diagram  $K \rightarrow \mathcal{P}_{\text{st},t}^{\text{L}}$  or  $K \rightarrow \mathcal{P}_{\text{st},t}^{\text{R}}$ ,  $(\varprojlim \mathcal{C}_k)^{\leq 0}$  (resp.  $(\varprojlim \mathcal{C}_k)^{\geq 0}$ ) is the full subcategory of  $\varprojlim \mathcal{C}_k$  spanned by objects whose image in  $\mathcal{C}_k$  is in  $\mathcal{C}_k^{\leq 0}$  (resp.  $\mathcal{C}_k^{\geq 0}$ ). For an interval  $I \subseteq \mathbb{Z}$ , we have an equivalence  $(\varprojlim \mathcal{C}_k)^{\in I} \rightarrow \varprojlim \mathcal{C}_k^{\in I}$ .

We denote by  $\mathcal{P}_{\text{st},t,\text{wrc}}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t,\text{rc,wlc}}^{\text{R}}$ ) the full subcategory of  $\mathcal{P}_{\text{st},t}^{\text{L}}$  (resp.  $\mathcal{P}_{\text{st},t}^{\text{R}}$ ) spanned by those  $\mathcal{C}$  that are weakly right complete (resp. right complete and weakly left complete). These full subcategories are stable under small limits in  $\mathcal{P}_{\text{st},t}^{\text{R}}$ .

**Proposition 4.1.7.** *Consider a diagram*

$$\begin{array}{ccc} (\mathcal{D}')^{\text{op}} & \xrightarrow{F} & \mathcal{P}_{\text{st},t,\text{rc,wlc}}^{\text{R}} \\ j^{\text{op}} \downarrow & & \downarrow P \\ \mathcal{D}^{\text{op}} & \xrightarrow{G} & \mathcal{Cat}_{\infty} \end{array}$$

of  $\infty$ -categories, where  $\mathcal{D}$  admits pullbacks,  $j$  is an inclusion satisfying the right lifting property with respect to  $\partial \Delta^n \subseteq \Delta^n$  for  $n \geq 2$ ,  $P$  is the forgetful functor. Assume that the arrows in  $\mathcal{D}'$  are stable under pullbacks in  $\mathcal{D}$  by arrows in  $\mathcal{D}'$ . Let  $X_{\bullet}^+ : N(\Delta_+)^{\text{op}} \rightarrow \mathcal{D}$  be a hypercovering for universal  $G$ -descent such that  $X_{\bullet}^+ | N(\Delta_{s+})^{\text{op}}$  factorizes through  $j$ . Then  $X_{\bullet}^+$  is an augmentation of  $G$ -descent.

*Proof.* By the right completeness of  $F(X_p^+)$  for  $p \geq -1$ , it suffices to show  $(F \circ (X_{\bullet}^+)^{\text{op}} | N(\Delta_{s+}))^{\leq 0}$  is a limit diagram. Let  $f_!$  be a left adjoint of the  $t$ -exact functor  $f^* : F(X_{-1}^+) \rightarrow \varprojlim F \circ (X_{\bullet}^+)^{\text{op}} | N(\Delta_s) = \mathcal{C}$ . The restrictions of these provide adjoint functors  $(f_!)^{\leq 0}$  and  $(f^*)^{\leq 0} : F(X_{-1}^+)^{\leq 0} \rightarrow \mathcal{C}^{\leq 0}$ . Let us first show that  $a : f_! f^* K \rightarrow K$  is an equivalence for all  $K \in F(X_{-1}^+)^{\leq 0}$ , namely that  $(f^*)^{\leq 0}$  is fully faithful. This is similar to Proposition 4.1.6. Let  $n \geq 0$ . The morphism  $h_{\bullet} : X_{\bullet}^+ \rightarrow \text{cosk}_n(X_{\bullet}^+ / X_{-1}^+) = Y_{\bullet}^+$  induces a diagram

$$\begin{array}{ccc} f_! f^* K & \xrightarrow{c} & g_! g^* K \\ & \searrow a & \swarrow b \\ & K & \end{array}$$

where  $g_!$  is a left adjoint of the  $t$ -exact functor  $g^* : F(X_{-1}^+) \rightarrow \varprojlim F \circ (Y_{\bullet}^+)^{\text{op}} | N(\Delta_s)$ . By Lemma 4.1.4,  $Y_{\bullet}^+$  is an augmentation of  $G$ -descent, so that  $b$  is an equivalence. Moreover,  $c = \varinjlim (f_p! f_p^* K \rightarrow$

$g_{p!}g_p^*K$ ), where  $f_{p!}$  is a left adjoint of  $f_p^*: F(X_{-1}^+) \rightarrow F(X_p^+)$ ,  $g_{p!}$  is a left adjoint of  $g_p^*: F(Y_{-1}^+) \rightarrow F(Y_p^+)$ ,  $f_{p!}f_p^* \rightarrow g_{p!}g_p^*$  is induced by  $h_p$ . By [20, 1.2.4.6, 1.2.4.7], we have a morphism of converging spectral sequences

$$\begin{array}{ccc} E_1^{pq} = H^q(f_{-p!}f_{-p}^*K) & \Longrightarrow & H^{p+q}f_!f^*K \\ c_1^{pq} \downarrow & & \downarrow H^{p+q}c \\ E_1'^{pq} = H^q(g_{-p!}g_{-p}^*K) & \Longrightarrow & H^{p+q}g_!g^*K, \end{array}$$

concentrated in the third quadrant. For  $p \geq -n$ , since  $h_p$  is an equivalence,  $c_1^{pq}$  is an isomorphism for all  $q$ . It follows that  $c_r^{pq}$  is an isomorphism for  $p+q \geq 1-n$ , and  $\tau^{\geq 1-n}c$  is an equivalence. Therefore,  $\tau^{\geq 1-n}a$  is an equivalence. Since  $n$  is arbitrary and  $F(X_{-1}^+)$  is weakly left complete,  $a$  is an equivalence.

It remains to show that  $d: L \rightarrow f^*f_!L$  is an equivalence for every  $L \in \mathcal{C}^{\leq 0}$ . Since  $\mathcal{C}$  is weakly left complete, it suffices to show that  $\tau^{\geq 1-n}d$  is an equivalence for every  $n \geq 1$ . For this, we may assume  $L \in \mathcal{C}^{[1-n, 0]}$ . We will show that  $L$  is in the essential image of  $(f^*)^{\leq 0}$ . Since  $(f^*)^{\leq 0}$  is fully faithful, this proves that  $d$  is an equivalence. Let  $H: \mathcal{P}r_{\text{st}, t, \text{rc}, \text{wlc}}^R \rightarrow \mathcal{C}at_n$  be the functor sending  $\mathcal{F}$  to  $\mathcal{F}^{[1-n, 0]}$ . It suffices to show that  $H \circ F \circ (X_\bullet^+)^{op} | N(\Delta_{s+})$  is a limit diagram. Since  $\mathcal{C}at_n$  is an  $(n+1)$ -category, we may assume that  $X_\bullet^+/X_{-1}^+$  is  $(n+1)$ -coskeletal by Lemma 4.1.5 applied to  $X_\bullet^+ \rightarrow \text{cosk}_{n+1}(X_\bullet^+/X_{-1}^+)$ . In this case,  $F \circ (X_\bullet^+)^{op} | N(\Delta_{s+})$  is a limit diagram by Lemma 4.1.4.  $\square$

The following variant of Proposition 4.1.7 will be used to establish proper hyperdescent. To state it conveniently, we introduce a bit of terminology. Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks,  $F: \mathcal{C}^{op} \rightarrow \mathcal{C}at_\infty$  be a functor. We say that a morphism  $f$  of  $\mathcal{C}$  is  $F$ -conservative if  $F(f)$  is conservative. We say that  $f$  is *universally  $F$ -conservative* if every pullback of  $f$  in  $\mathcal{C}$  is  $F$ -conservative. We say that an augmented simplicial object  $X_\bullet^+$  of  $\mathcal{C}$  is a *hypercovering for universal  $F$ -conservativeness* if  $X_n^+ \rightarrow (\text{cosk}_{n-1}(X_\bullet^+/X_{-1}^+))_n$  is universally  $F$ -conservative for all  $n \geq 0$ .

**Proposition 4.1.8.** *Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks, let  $F: \mathcal{C}^{op} \rightarrow \mathcal{P}r_{\text{st}, t, \text{wrc}}^L$  be a functor, let  $a$  be an integer, let  $G: \mathcal{P}r_{\text{st}, t, \text{wrc}}^L \rightarrow \mathcal{C}at_\infty$  be the functor sending  $\mathcal{C}$  to  $\mathcal{C}^{\geq a}$  (resp.  $\mathcal{C}^+ = \bigcup_n \mathcal{C}^{\geq n}$ ), and let  $X_\bullet^+$  be a hypercovering for universal  $(G \circ F)$ -descent (resp. and universal  $(P \circ F)$ -conservativeness, where  $P: \mathcal{P}r_{\text{st}, t, \text{wrc}}^L \rightarrow \mathcal{C}at_\infty$  is the forgetful functor). Then  $X_\bullet^+$  is an augmentation of  $(G \circ F)$ -descent.*

*Proof.* The proof of the case of  $\mathcal{C}^{\geq a}$  is similar to the proof of Proposition 4.1.7. In the case of  $\mathcal{C}^+$ , the conservativeness implies that  $G(\varprojlim F \circ (X_\bullet^+)^{op}) \rightarrow \varprojlim G \circ F \circ (X_\bullet^+)^{op}$  is an equivalence. The rest of the proof is similar.  $\square$

Consider the functors [18, §6.4]

$$\begin{aligned} \text{Chp}^{\text{Ar}} \text{EO}_\otimes^*: N(\text{Chp}^{\text{Ar}})^{op} &\rightarrow \text{Fun}(N(\mathcal{R}\text{ind})^{op}, \mathcal{P}r_{\text{st}, \text{cl}}^{L\otimes}), \\ \text{Chp}_L^{\text{Ar}} \text{EO}_!^*: N(\text{Chp}_L^{\text{Ar}})^{op} &\rightarrow \text{Fun}(N(\mathcal{R}\text{ind}_{L\text{-tor}})^{op}, \mathcal{P}r_{\text{st}}^L), \end{aligned}$$

and their adic version in §1.4

$$\begin{aligned} \text{Chp}^{\text{Ar}} \underline{\text{EO}}_\otimes^*: N(\text{Chp}^{\text{Ar}})^{op} &\rightarrow \text{Fun}(N(\mathcal{R}\text{ind})^{op}, \mathcal{P}r_{\text{st}, \text{cl}}^{L\otimes}), \\ \text{Chp}_L^{\text{Ar}} \underline{\text{EO}}_!^*: N(\text{Chp}_L^{\text{Ar}})^{op} &\rightarrow \text{Fun}(N(\mathcal{R}\text{ind}_{L\text{-tor}})^{op}, \mathcal{P}r_{\text{st}}^L). \end{aligned}$$

We say that an augmented simplicial object  $X_\bullet^+$  in  $\text{Chp}^{\text{Ar}}$  (or  $\text{Chp}^{\infty\text{-DM}}$  introduced below) is a  $(P)$  *hypercovering* for a property  $(P)$  on morphisms if  $X_q^+ \rightarrow (\text{cosk}_{q-1}(X_\bullet^+/X_{-1}^+))_q$  is *surjective* and satisfies  $(P)$  for all  $q \geq 0$ .

**Proposition 4.1.9.** *Every smooth hypercovering in  $\mathrm{Chp}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}$ ) is an augmentation of  ${}_{\mathrm{Chp}^{\mathrm{Ar}}}\mathrm{EO}_{\otimes}^*$ -descent (resp.  ${}_{\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}}\mathrm{EO}_{!}^{\mathrm{op}}$ -descent) and  ${}_{\mathrm{Chp}^{\mathrm{Ar}}}\mathrm{EO}_{\otimes}^*$ -descent (resp.  ${}_{\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}}\mathrm{EO}_{!}^{\mathrm{op}}$ -descent).*

Together with [20, 1.2.4.5] and its dual version, Proposition 4.1.9 implies Theorem 0.1.3.

*Proof.* Let  $X_{\bullet}^+$  be an augmented simplicial object of  $\mathrm{Chp}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}$ ). It suffices to apply Proposition 4.1.7 to the full subcategory  $\mathrm{Chp}_{\mathrm{sm}/X_{-1}}^{\mathrm{Ar}} \subseteq \mathrm{Chp}_{/X_{-1}}^{\mathrm{Ar}}$  spanned by higher Artin stacks smooth over  $X_{-1}$ . In the notation of Proposition 4.1.7,  $F$  associates the usual  $t$ -structure (resp. the usual  $t$ -structure shifted by twice the relative dimension over  $X_{-1}$ ). This proof applies to both the non-adic case and the adic case. The adic case can also be deduced from the non-adic case by taking limits.  $\square$

**Definition 4.1.10.** The  $\infty$ -category of  $\infty$ -DM stacks  $\mathrm{Chp}^{\infty\text{-DM}}$  is the  $\infty$ -category  $\mathrm{Sch}(\mathcal{G}_{\mathrm{et}}(\mathbb{Z}))$  of  $\mathcal{G}_{\mathrm{et}}(\mathbb{Z})$ -schemes in the sense of [21, 2.3.9, 2.6.11].

Using Proposition 4.1.7, we can adapt the DESCENT program [18, §4] to define enhanced operation maps for  $\infty$ -DM stacks:

$$\begin{aligned} {}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO} &: \delta_{2,\{2\}}^* \mathrm{Fun}(\Delta^1, \mathrm{Chp}^{\infty\text{-DM}})_{F^0, A \rightarrow}^{\mathrm{cart}} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathrm{Cat}_{\infty})), \\ {}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{\otimes}^* &: (\mathrm{Chp}^{\infty\text{-DM}})^{\mathrm{op}} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind})^{\mathrm{op}}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L}\otimes}), \end{aligned}$$

compatible with the enhanced operation maps constructed for higher DM stacks [18, §6.5]. Applying Definition 1.4.1, we obtain similarly the enhanced adic operation maps for  $\infty$ -DM stacks:

$$\begin{aligned} {}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO} &: \delta_{2,\{2\}}^* \mathrm{Fun}(\Delta^1, \mathrm{Chp}^{\infty\text{-DM}})_{F^0, A \rightarrow}^{\mathrm{cart}} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{P}\mathrm{f}}^{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}}(\mathrm{Cat}_{\infty})), \\ {}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{\otimes}^* &: (\mathrm{Chp}^{\infty\text{-DM}})^{\mathrm{op}} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind})^{\mathrm{op}}, \mathcal{P}_{\mathrm{st}, \mathrm{cl}}^{\mathrm{L}\otimes}), \end{aligned}$$

compatible with the enhanced adic operation maps constructed for higher DM stacks in §1.4. By restriction, we obtain functors:

$${}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{!}, {}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{!}^*: (\mathrm{Chp}^{\infty\text{-DM}})_F \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}}, \mathcal{P}_{\mathrm{st}}^{\mathrm{L}}).$$

We obtain the following result.

**Proposition 4.1.11.** *Every smooth hypercovering in  $\mathrm{Chp}^{\infty\text{-DM}}$  is an augmentation of  ${}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{\otimes}^*$ -descent and (resp.  ${}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{!}^{\mathrm{op}}$ -descent) and  ${}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{\otimes}^*$ -descent (resp.  ${}_{\mathrm{Chp}^{\infty\text{-DM}}}\mathrm{EO}_{!}^{\mathrm{op}}$ -descent).*

**4.2. Proper descent.** The following lemma is an immediate consequence of [20, Proposition 1.2.4.5].

**Lemma 4.2.1.** *Let  $\mathcal{C}, \mathcal{D}$  be stable  $\infty$ -categories equipped with left complete  $t$ -structures. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a conservative  $t$ -exact functor. Then  $\mathcal{C}^{\leq 0}$  admits geometric realizations, and geometric realizations are preserved by  $F$ .*

**Lemma 4.2.2.** *Let  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  be stable  $\infty$ -categories equipped with  $t$ -structures such that  $\mathcal{C}$  and  $\mathcal{D}$  are left complete and right complete. Let  $F: \mathcal{C} \rightarrow \mathcal{D}, G: \mathcal{C} \rightarrow \mathcal{E}$  be conservative  $t$ -exact functors. Then  $\mathcal{C}$  admits  $G$ -split geometric realizations, and those geometric realizations are preserved by  $F$ .*

*Proof.* Let  $X_{\bullet}$  be a  $G$ -split simplicial object of  $\mathcal{C}$ ,  $Y_{\bullet}: \mathrm{N}(\Delta_{-\infty})^{\mathrm{op}} \rightarrow \mathcal{D}$  be an extension of  $G \circ X_{\bullet}$ . Then the unnormalized cochain complex

$$\cdots \rightarrow H^q Y_2 \rightarrow H^q Y_1 \rightarrow H^q Y_0 \rightarrow H^q Y_{-1} \rightarrow 0$$

is acyclic. It follows that the unnormalized cochain complex

$$\cdots \rightarrow H^q X_2 \rightarrow H^q X_1 \xrightarrow{\theta^q} H^q X_0$$

is an acyclic resolution of the object  $A^q = \mathrm{coker}(\theta^q)$  in the heart of  $\mathcal{C}$ . The lemma then follows from [20, Corollary 1.2.4.12].  $\square$

Consider the functors

$$\begin{aligned} \mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}} \mathrm{EO}^* &: \mathrm{N}(\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}})^{op} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{L-tor}})^{op}, \mathcal{C}\mathrm{at}_{\infty}), \\ \mathrm{Chp}^{\infty\text{-DM}}_{\mathrm{tor}} \mathrm{EO}^* &: \mathrm{N}(\mathrm{Chp}^{\infty\text{-DM}})^{op} \rightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{op}, \mathcal{C}\mathrm{at}_{\infty}) \end{aligned}$$

sending  $X$  to  $\lambda \mapsto \mathcal{D}^{\geq 0}(X, \lambda)$ . Note that here we have put extra conditions on the categories of ringed diagrams.

**Proposition 4.2.3.** *Let  $\mathcal{S}$  be an  $\mathrm{L}$ -coprime locally Noetherian higher Artin stack (that is, there exists an atlas  $S \rightarrow \mathcal{S}$  where  $S$  is a locally Noetherian scheme).*

(1) *For every object  $\lambda$  of  $\mathcal{R}\mathrm{ind}_{\mathrm{L-tor}}$  and every Cartesian square*

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

*in  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}_{\mathrm{Lft}/\mathcal{S}}^{\mathrm{Ar}}$ ) with  $p$  proper of finite diagonal (resp. proper, 1-Artin, of separated diagonal), the induced square*

$$\begin{array}{ccc} \mathcal{D}^{\geq 0}(Z, \lambda) & \xleftarrow{p^*} & \mathcal{D}^{\geq 0}(X, \lambda) \\ g^* \downarrow & & \downarrow f^* \\ \mathcal{D}^{\geq 0}(W, \lambda) & \xleftarrow{q^*} & \mathcal{D}^{\geq 0}(Y, \lambda) \end{array}$$

*is right adjointable.*

(2) *Every proper finite-diagonal hypercovering in  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}}$  (resp. proper, 1-Artin, separated-diagonal hypercovering in  $\mathrm{Chp}_{\mathrm{Lft}/\mathcal{S}}^{\mathrm{Ar}}$ ) is an augmentation of  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}} \mathrm{EO}^*$ -descent.*

A similar result holds for  $\mathrm{Chp}^{\infty\text{-DM}}_{\mathrm{tor}} \mathrm{EO}^*$ .

*Proof.* Let us first show that (1) implies (2). By Proposition 4.1.8, to show (2), it suffices to show that every surjective morphism proper of finite diagonal (resp. proper, 1-Artin, of separated diagonal) is of  $\mathrm{Chp}_{\mathrm{L}}^{\mathrm{Ar}} \mathrm{EO}^*$ -descent. For this, we apply [18, 3.3.6]. Assumption (1) of [18, 3.3.6] follows from the dual of Lemma 4.2.1. Assumption (2) of [18, 3.3.6] is (1).

To show (1), applying [18, 4.3.6] and smooth base change, we are reduced to the case where  $X$  and  $Y$  are quasi-compact quasi-separated schemes. In this case, there exists a finite cover [25, Theorem B] (resp. proper cover [22, 1.1])  $r_0: X_0 \rightarrow X$ . Since (1) is known in the case of algebraic spaces,  $r_0$  is of descent, so that every object of  $\mathcal{D}^{\geq 0}(X, \lambda)$  has the form  $\varprojlim_{n \in \Delta} r_{n*} r_n^* \mathcal{K}$ , where  $r_{\bullet}$  is the Čech nerve of  $r_0$ . We then conclude by Lemma 4.2.1.  $\square$

The above result can be extended to  $\mathcal{D}(X, \lambda)$  under cohomological finiteness conditions. For an object  $\lambda$  of  $\mathcal{R}\mathrm{ind}$ , Consider the functors

$$\begin{aligned} \mathrm{Chp}^{\mathrm{Ar}}_{\lambda} \mathrm{EO}_{\otimes}^* &: \mathrm{N}(\mathrm{Chp}^{\mathrm{Ar}})^{op} \rightarrow \mathcal{C}\mathrm{at}_{\infty}, \\ \mathrm{Chp}^{\infty\text{-DM}}_{\lambda} \mathrm{EO}_{\otimes}^* &: \mathrm{N}(\mathrm{Chp}^{\infty\text{-DM}})^{op} \rightarrow \mathcal{C}\mathrm{at}_{\infty} \end{aligned}$$

sending  $X$  to  $\lambda \mapsto \mathcal{D}(X, \lambda)^{\otimes}$ .

**Proposition 4.2.4.** *Let  $\mathcal{S}$  be an  $\mathrm{L}$ -coprime locally Noetherian higher Artin stack (that is, there exists an atlas  $S \rightarrow \mathcal{S}$  where  $S$  is a locally Noetherian scheme).*

- (1) Let  $\lambda$  be an object of  $\mathcal{R}\text{ind}_{\mathcal{L}\text{-tor}}$  and consider a Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

in  $\text{Chp}_{\mathcal{L}}^{\text{Ar}}$  (resp.  $\text{Chp}_{\text{ft}/S}^{\text{Ar}}$ ) with  $p$  proper of finite diagonal (resp. proper, 1-Artin, of separated diagonal). Assume that for every morphism  $U \rightarrow X$  locally of finite type with  $U$  an affine scheme,  $X_0$  is  $\lambda$ -cohomologically finite. Then the induced square

$$\begin{array}{ccc} \mathcal{D}(Z, \lambda) & \xleftarrow{p^*} & \mathcal{D}(X, \lambda) \\ g^* \downarrow & & \downarrow f^* \\ \mathcal{D}(W, \lambda) & \xleftarrow{q^*} & \mathcal{D}(Y, \lambda) \end{array}$$

is right adjointable.

- (2) Let  $X_{\bullet}^+$  be a proper finite-diagonal hypercovering in  $\text{Chp}_{\mathcal{L}}^{\text{Ar}}$  (resp. proper, 1-Artin, separated-diagonal hypercovering in  $\text{Chp}_{\text{ft}/S}^{\text{Ar}}$ ). Assume that for every morphism  $U \rightarrow X_{-1}^+$  locally of finite type with  $U$  an affine scheme,  $X_0$  is  $\lambda$ -cohomologically finite. Then  $X_{\bullet}^+$  is an augmentation of  ${}_{\text{Chp}^{\text{Ar}}, \lambda} \text{EO}_{\otimes}^*$ -descent.

A similar result holds for  ${}_{\text{Chp}^{\infty\text{-DM}}, \lambda} \text{EO}_{\otimes}^*$  for  $\lambda$  in  $\mathcal{R}\text{ind}_{\text{tor}}$ .

*Proof.* It suffices to repeat the proof of Proposition 4.2.3 with Proposition 4.1.8 replaced by Proposition 4.1.7 and Lemma 4.2.1 replaced by Lemma 4.2.2.  $\square$

**4.3. Flat descent.** Consider the functors

$$\begin{aligned} {}_{\text{Chp}^{\text{Ar}}} \text{EO}^* &: \mathcal{N}(\text{Chp}^{\text{Ar}})^{op} \rightarrow \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Cat}_{\infty}), \\ {}_{\text{Chp}^{\infty\text{-DM}}} \text{EO}^* &: \mathcal{N}(\text{Chp}^{\infty\text{-DM}})^{op} \rightarrow \text{Fun}(\mathcal{N}(\mathcal{R}\text{ind})^{op}, \text{Cat}_{\infty}) \end{aligned}$$

sending  $X$  to  $\lambda \mapsto \mathcal{D}^{\geq 0}(X, \lambda)$ .

**Proposition 4.3.1.** Any flat and locally finitely presented hypercovering of higher Artin stacks (resp.  $\infty$ -DM stacks) is an augmentation of  ${}_{\text{Chp}^{\text{Ar}}} \text{EO}^*$ -descent (resp.  ${}_{\text{Chp}^{\infty\text{-DM}}} \text{EO}^*$ -descent).

This is an analogue of flat cohomological descent [2, Vbis Proposition 4.3.3 c)].

*Proof.* By Proposition 4.1.8, we are reduced to show that any surjective flat and locally finitely presented morphism  $f: Y \rightarrow X$  of higher Artin stacks (resp.  $\infty$ -DM stacks) is of descent. By [18, 3.1.2] and smooth (resp. étale) descent, we are reduced to the case of schemes. Let  $X'$  be a disjoint union of strict localizations of  $X$ , such that the morphism  $g: X' \rightarrow X$  is surjective. By [1, 17.16.2, 18.5.11], there exists a finite surjective morphism  $g': Z \rightarrow X'$ , such that the composite map  $Z \rightarrow X$  factorizes through  $f$ . Since  $g$  and  $g'$  are both of universal  ${}_{\text{sch}}^{\geq 0} \text{EO}^*$ -descent by [18, 3.3.6] and the dual of Lemma 4.2.1, it follows from [18, 3.1.2] that  $f$  is of universal  ${}_{\text{sch}}^{\geq 0} \text{EO}^*$ -descent.  $\square$

Again this result can be extended to  $\mathcal{D}(X, \lambda)$  under cohomological finiteness conditions.

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